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DETERMINISTIC METHODS IN STOCHASTIC OPTIMAL CONTROL

by

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Abstract

In this research a new approach to the control of systems represented by stochastic differential equations (SDEs) is developed in which stochastic control is viewed as deterministic control with a particular form of constraint structure. Specifically, the characteristic "non-anticipativity" property of the control processes is formulated as an equality constraint on the set of possibly anticipative processes. The optimal non-anticipative control is then recovered by minimizing, over the class of possibly anticipating processes, a cost function modified by the inclusion of a Lagrange multiplier term to enforce the nonanticipativity constraint. This unconstrained minimization is carried out "pathwise" i.e. separately for each value of the random parameter ω and hence reduces to a parametrized family of deterministic optimal control problems.

Solutions of the controlled SDEs with anticipative controls are defined by a decomposition method. It is shown that the value function of the control problem is the unique global solution of a robust equation (random partial differential equation) associated to a linear backward Hamilton-Jacobi-Bellman stochastic partial differential equation (HJB SPDE). This appears as limiting SPDE for a sequence of random HJB PDE's when linear interpolation approximation of the Wiener process is used. Our approach extends the Wong-Zakai type results [20] from SDE to the stochastic dynamic programming equation by showing how this arises as average of the limit of a sequence of deterministic dynamic programming equations. The stochastic characteristic method of Kunita [13] is used to represent the value function. By choosing the Lagrange multiplier equal to its nonanticipative constraint value the usual stochastic (nonanticipative) optimal control and optimal cost are recovered. The anticipative optimal control problem is formulated and solved by almost sure deterministic optimal control. We obtain a PDE for the "cost of perfect information": the difference between the cost function of the nonanticipative control problem and the cost of the anticipative problem which satisfies a nonlinear backward HJB SPDE. Poisson bracket conditions are found ensuring this has a global solution. The cost of perfect information is shown to be zero when a Lagrangian submanifold is invariant for the stochastic characteristics. LQG and a nonlinear anticipative control problem are considered as examples in this framework .

KEYWORDS : stochastic control, optimal control, dynamic programming, cost of information, stochastic partial differential equations, stochastic flows, anticipative stochastic differential equations

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0 : Introduction and presentation of the report

At first sight it appears that stochastic control is simply a generalization of deterministic control. One sees this for example in the familiar LQG problem: given the solution to the stochastic linear regulator problem (with white noise perturbations) one obtains the solution for the deterministic case simply by setting the noise mean and covariance to zero. This is, however, a misleading example and, in general, deterministic optimal control is far from being a trivial by-product of stochastic control, which is in fact in some respects substantially simpler. Indeed, much of it concerns the "uniformly elliptic" case, where the "smoothing" properties of Brownian motion make dynamic programming in its simplest form a viable technique and obviate the need for special methods to handle non-differentiability. In this research an alternative approach is developed, in which stochastic control is viewed as deterministic control with a particular form of constraint structure. Thus the distinction between "deterministic" and "stochastic" is in principle eradicated. (In practice, of course, it is not, since the form of the constraints leads to idiosyncratic solution techniques.) The first to espouse this point of view were Rockafellar and Wets [16], and a succession of papers followed in the stochastic programming literature, including some very general formulations in continuous time. The term *stochastic programming* denotes a problem in which the cost or reward is given as some explicit function of a decision process and some random parameters. Until recently little had been done in this vein in stochastic control, i.e. problems where the cost is determined implicitly through the evolution of some dynamical system, mainly because of apparent difficulties in formulating the problem correctly. These difficulties have now been overcome, and the Rockafellar/Wets approach has been explored with some thoroughness.

Stochastic optimization problems typically take the form

$$\min_{d \in \mathfrak{D}} EJ(d, \omega)$$

where $\omega \in \Omega$, the set of random events, $\mathfrak{D}$ is some class of functions $d:\Omega \to F$ (F is some space of decisions) and E denotes expectation with respect to a probability P on Ω. Note that this problem is truly "stochastic" only when ω is incompletely known. If the controller knows ω in advance then (ignoring technicalities) $\mathfrak{D}$ is the set of *all* functions $d:\Omega \to F$ and we have

$$\min_{d \in \mathfrak{D}} EJ(d,\omega) = E \min_{f \in F} J(f,\omega).$$

Thus minimization can be carried out separately for each ω and the only role of the probability P is to average the result. In dynamic optimization F is typically a class of functions such as $L_\infty[T;U]$, U being the "action space" and T a time set, so that the controls are stochastic processes $u(t,\omega)$. The most basic requirement is that these processes respect the flow of information, i.e. depend at each time only on what has been observed up to that time. In fact, this is a *linear equality constraint*. To see this in the simplest setting, suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, U = \mathbf{R}, T = \{0,1\}$. A control $u(t,\omega)$ is then equivalent to an 8-vector $x = (x_1, \ldots, x_8)$ where

$$x_i = u_0(\omega_i), \quad i = 1 \ldots 4$$

$$x_i = u_1(\omega_{i-4}), \quad i = 5 \ldots 8$$

and the cost can be expressed as

$$EJ(u,\omega) = \sum_{i=1}^{4} p_i J(u_0(\omega_i), u_1(\omega_i), \omega_i) \quad =: g(x)$$

where $p_i = P(\{\omega_i\})$. Let $A = \{\omega_1, \omega_2\}$ and suppose that at time 0 we have no observations while at time 1 we discover whether A or A^c has occurred. The decision rule must then be a constant at time 0 and constant on A and A^c at time 1. But this means that x must satisfy the equality constraint $Hx = 0$, where

$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Or, put another way, x must lie in a certain 3-dimensional subspace S of $\mathbf{R}^8$. There is therefore a Lagrange multiplier, i.e. a 5-vector $\lambda \in S^{\perp}$ such that the optimal decision is a global minimum of $g(x) + \lambda^T Hx$. The multiplier λ is the *price for information* in that it gives the incremental decrease in cost that is available ifthe information constraint is waived.

In this research these ideas are extended to continuous-time dynamical systems, i.e. to the problem of minimizing

$$E[\Theta(x_T)]$$

where the process x_t satisfies the controlled stochastic differential equation

$$dx_t = f(x_t, u_t)dt + g(x_t)dw_t.$$

Here w_t is a Brownian motion and u_t is the (non-anticipative) control process.

In the mid 60's Wong and Zakai [20] addressed the problems of approximating stochastic integrals by ordinary ones and approximating the solutions of stochastic differential equations (SDE) by the solutions of sequences of ordinary differential equations using piecewise linear approximations of the Wiener process. Sussmann [18] continued in the 70's by studying when can the solution of a SDE be defined by "extending by continuity to C^0" noises the solution of a C^1 noise-driven ordinary differential equation. Using these ideas, Davis [5] reduced the LQG problem for the linear controlled diffusions driven by Wiener processes to a family of deterministic linear quadratic optimal control problems (parametrized by the paths of the driving Wiener process) with nonanticipativity equality constraint on the set of admissible possibly anticipative controls. First, a pathwise optimal cost is deterministically evaluated for each of the problems of the family by first considering driving stochastic processes with C^1 paths and then using the work of Sussmann [18] to extend the cost to C^0 paths by continuity . Finally, averaging over the sample space yields the usual LQG optimal cost.

The report will approach in this perspective the relation between *nonlinear* deterministic and stochastic control problems by reducing the nonlinear stochastic optimal control problem to a family of deterministic optimal control problems with a cost containing a nonanticipativity constraint. Consider the nonlinear (nonanticipative) stochastic optimal control problem

$$dx_t = f(x_t, u_t)\,dt + g(x_t)\,dw_t \tag{0.1}$$

$$x_0 = x(0)$$

$$\inf_{u \in \mathcal{N}} E\,[\theta(x(T))] := \inf_{u \in \mathcal{N}} E\,\mathfrak{J}(u,w) \tag{0.2}$$

where $\mathcal{N}$ is the class of nonanticipative (adapted) controls with values in a compact set $\mathcal{U} \subset \mathbb{R}^m$ ($\mathcal{N}$ will be defined in detail in the next section). If we can define $\mathfrak{J}(u,w)$ for non-adapted (anticipative) controls $u \in \mathcal{A}$ with values in the compact set $\mathcal{U} \subset \mathbb{R}^m$ (see next section for the complete definition of $\mathcal{A}$) then

$$\inf_{u \in \mathcal{A}} E\,\mathfrak{J}(u,w) = E \inf_{u_\omega \in \mathcal{M}} \mathfrak{J}(u_\omega, w(\cdot,\omega)) \tag{0.3}$$

provided we make some assumptions ensuring that the infimum on the right is attained for each ω and the function assigning to each ω the corresponding minimizing control function is measurable. $\mathcal{M}$ is the class of measurable $\mathcal{U}$-valued (deterministic) controls. This requires of course a definition for the solution of an anticipative SDE (with anticipative drift) which will be done in Section 1 using the decomposition of solutions of SDE's (see Kunita [14, p. 268] and Ocone and Pardoux [15]). (0.3) can be used to solve problem (0.2) by solving a family of deterministic optimal control problems indexed by $\omega \in \Omega$ for a wider class of controls $u \in \mathcal{A}$. This can be done by adding to the cost a Lagrange multiplier corresponding to the nonanticipativity constraint as a linear functional of the controls and by solving

$$\inf_{u_\omega \in \mathcal{M}} [\mathcal{J}(u_\omega, w(.,\omega)) + <\lambda(\omega), u_\omega)>]$$

for each $\omega \in \Omega$ and then averaging over Ω. To recover (0.2) we would like the above minimum to be attained at the same $u(., \omega)$ at which (0.2) is minimized. Assuming (0.2) has an optimal feedback solution $u(t, x_t)$, we will consider the nonanticipativity constraint as an integral cost term

$$\int_0^T \lambda^T(t, x_t, \omega)\, u_\omega(t)\, dt$$

and we will define $\lambda^T(t, x, \omega)$ (superscript T denotes transpose) so that $\lambda^T(t, x_t, \omega)$ is L^1

integrable (i.e. $E\int_0^T |\lambda^T(t, x_t, \omega)|\, dt < \infty$ for any $u_t \in \mathcal{A}$ where x_t is the solution of (0.1)

corresponding to u_t) and :

(i) $$\arg\inf_{u_\omega \in \mathcal{M}} [\mathcal{J}(u_\omega, w(., \omega)) + \int_0^T \lambda^T(t, x(t, \omega), \omega)\, u_\omega(t) dt] =$$

$$= \arg\inf_{u \in \mathcal{N}} \mathcal{J}(u, w) = u^*(t, x(t, \omega))$$

(ii) $E\, \lambda\, (t, x, \omega) = 0$

(iii) $\lambda(t, x, \omega)$ is $\mathcal{F}_t^T$ - adapted (i.e. future adapted : $\mathcal{F}_t^T$ is the sigma algebra generated by the future increments of the driving Wiener process $w(\tau, \omega) - w(s, \omega)$; $t \leq \tau \leq s \leq T$)

Conditions (ii) and (iii) imply

$$E\int_0^T \lambda^T(t, x(t, \omega), \omega)\, u(t, \omega)\, dt = 0$$

for nonanticipative controls $u(...) \in \mathcal{N}$.

Our approach extends the Wong-Zakai type of results [20] (which show how solutions of stochastic differential equations arise as limits of solutions of sequences of ordinary differential equations) from SDE to optimal control problems for controlled SDE via the HJB equation of dynamic programming. In the case when nonanticipating controls appear in the drift the Wong-Zakai convergence result states that under smoothness and boundedness assumptions on the coefficients we have ($P\text{-}\lim_{n\to\infty}$ denotes limit in probability)

$$P\text{-}\lim_{n\to\infty} x^n(t,\omega) = x(t,\omega)$$

where

$$\frac{dx^n(t,\omega)}{dt} = \overline{f}(x^n(t,\omega),u(t,\omega)) + g(x^n(t,\omega))\,\frac{dr^n(t,\omega)}{dt} \;;\; x^n(0,\omega)= x_0 \tag{0.4}$$

with

$$r^n(t,\omega) = w(t_k,\omega) + 2^n(t - t_k)\,(w(t_{k+1},\omega) - w(t_k,\omega));\ t \in [t_k, t_{k+1}) \tag{0.5}$$

$$t_k = k/2^n;\ k = 0, 1, \dots, [T/2^n]$$

$$\frac{dr^n(t,\omega)}{dt} = 2^n\,(w(t_{k+1},\omega) - w(t_k,\omega)) \ \text{ for } t \in [t_k, t_{k+1}]$$

and

$$dx(t,\omega) = \overline{f}(x(t,\omega),u(t,\omega))\,dt + g(x(t,\omega))\circ dw(t,\omega)\;;\; x(0,\omega) = x_0 \tag{0.6}$$

$$\overline{f}(x,u) = f(x,u) - \tfrac{1}{2}\,g_x g(x)$$

where "$\circ$" denotes Stratonovich differential and $u(t,\omega) \in \mathcal{N}$ are the paths of the nonanticipative control. As it is known (0.6) can be written in Ito form as (0.1) by adding a correction term to the drift. Let us consider the stochastic optimal control problem (0.1),(0.2). The dynamic programming second order PDE of stochastic optimal control [8, p.154] gives under suitable assumptions the value function $V(t,x) := \inf_{u\in\mathcal{N}} E[\theta(x(T;t,x,\omega))]$ ($x(T;t,x,\omega)$ is the solution of (0.1) at time T started at time t from x) :

$$\frac{\partial V(t,x)}{\partial t} + \min_{u \in \mathcal{U}} \{\frac{\partial V}{\partial x}(t,x) f(x,u)\} + \frac{1}{2}\mathrm{tr}\,(\frac{\partial^2 V}{\partial x^2} gg^T) = 0 : V(T,x) = \theta(x) \tag{0.7}$$

This is shown by using the probabilistic representation formula for the solution of (0.7). Let us consider now the sequence of pathwise deterministic optimal control problems for (0.4) :

$$\inf_{u \in \mathcal{M}} [\theta(x^n(T,\omega)) + \int_0^T (\lambda^n(t,x^n(t,\omega),\omega))^T u(t,\omega)dt\ ] \tag{0.8}$$

where λ^n is the approximation sequence (corresponding to (0.5)) of the Lagrange multiplier process allowing us to solve (0.2) over the enlarged class of possibly anticipating controls by solving the family of pathwise optimal control problems over $u \in \mathcal{M}$ for almost all $\omega \in \Omega$

$$\inf_{u \in \mathcal{M}} [\theta(x(T,\omega)) + \int_0^T \lambda^T(t,x(t,\omega),\omega)u(t,\omega)dt]$$

The value function $V^n(t,x,\omega) := \inf_{u \in \mathcal{M}} [\theta(x^n(T;t,x,\omega)) + \int_0^T (\lambda^n(\tau,x^n(\tau;t,x,\omega),\omega))^T u(\tau,\omega)d\tau\ ]$

of (0.8) satisfies the sequence of families of dynamic programming first order PDEs (parametrized by ω) of deterministic optimal control :

$$\frac{\partial V^n}{\partial t}(t,x,\omega) + \min_{u \in \mathcal{U}} \{ \frac{\partial V^n}{\partial x}(t,x,\omega)\,\tilde{f}(x,u) + \lambda(t,x,\omega)u\} + \frac{\partial V^n}{\partial x}(t,x,\omega)g(x)\,\frac{dw^n(t,\omega)}{dt} = 0$$

$$V^n(T,x,\omega) = \theta(x) \tag{0.9}$$

The Lagrange multiplier process that will be introduced in the report will give an answer to the intriguing question : How can we arrive at (0.7) from (0.9) namely what is the "bridge" between the second order PDE of stochastic dynamic programming and the first order PDE of deterministic dynamic programming ? The answer is

$$V(t,x) = E\ P\text{-}\lim_{n \to \infty} V^n(t,x,\omega)$$

and the bridge itself which fills the "gap"(in the terminology of [18]) between stochastic and deterministic optimal control is a new Hamilton-Jacobi-Bellman backward Stratonovich stochastic partial differential equation that will be introduced and studied by us

$$d\overline{V}(t, x,\omega) + \min_{u \in \mathcal{U}} \{ \frac{\partial \overline{V}}{\partial x} (t, x,\omega)\, \overline{f}(x, u) + \lambda(t,x,\omega)u\}\, dt + \frac{\partial \overline{V}}{\partial x} (t, x,\omega)\, g(x) \,\hat{\circ}\, dw_t = 0$$

$$\overline{V}(T, x,\omega) = \theta(x) \tag{0.10}$$

having the unique global solution $\overline{V}(t,x,\omega) = P\text{-}\lim_{n\to\infty} V^n(t,x,\omega)$ in a certain space of Hölder semimartingales to be defined precisely in Section 1. Our deterministic methods will concentrate on the robust PDE associated to (0.10) defined almost surely

$$\frac{\partial W}{\partial t} (t, \eta,\omega) + \min_{u \in \mathcal{U}} \{ \frac{\partial W}{\partial \eta} (t, \eta,\omega) \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\, \overline{f}\, (\xi_t(\eta),u) + \lambda(t,\xi_t(\eta),\omega)u\} = 0 \tag{0.11}$$

$$W(T, \eta,\omega) = \theta \circ \xi_T(\eta)$$

where it is proved that $W(t,\xi_t^{-1}(x),\omega) = \overline{V}(t,x,\omega)$. We use the almost sure decomposition of the stochastic flow of (0.1) [14] given by

$$x_t(x) = \xi_t \circ \eta_t(x)$$

$$d\xi_t(y) = g(\xi_t(y)) \circ dw_t,\ \xi_0(y) = y \tag{0.12}$$

$$\frac{d\eta_t(x)}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t(x))\, \overline{f}(\xi_t \circ \eta_t(x), u_t)\ ;\ \eta_0(x) = x(0) \tag{0.12'}$$

This decomposition holds on some $\Omega' \subset \Omega$ with $P(\Omega')=1$ for which $\xi_t(y)$ is a global flow of diffeomorphisms as we will see later on. The robust PDE (0.11) is the dynamic programming equation for the pathwise optimal control problems

$$\inf_{u \in \mathcal{M}} [\theta \circ \xi_T(\eta_T) + \int_0^T \lambda(t,\xi_t(\eta_t),\omega)u(t,\omega)dt]$$

subject to (0.12') as only this involves controls.

We will also consider the anticipative optimal control problem ($\lambda=0$) which will be solved by reduction to pathwise deterministic control via the stochastic flow decomposition formula. The problem in this case is that the random value function of the family is characterized by a nonliner backward SPDE which does not always have a global stochastic characteristics solution. We give conditions ensuring this by turning a Lagrangian submanifold (see the stochastic mechanics of Bismut [3], Arnold [1]) into an invariant manifold for the stochastic hamiltonian system of characteristic equations. The optimal control is given by a selection lemma and a formula is obtained for the cost of information on the future (i.e. the difference between the nonanticipative and anticipative value functions) :

$$\Delta(t,x) = V(t,x) - EV^0(t,x,\omega)$$

where as we will see $V^0(t,x,\omega) = \inf_{u \in \mathcal{M}} \theta(x(T;t,x,\omega))$ satisfies the HJB SPDE

$$dV^0(t,x,\omega) + \min_{u \in \mathcal{U}} \{\frac{\partial V^0}{\partial x}(t,x,\omega)\tilde{f}(x,u)\}dt + \frac{\partial V^0}{\partial x}(t,x,\omega)g(x)\hat{o}\, dw_t = 0 \qquad (0.13)$$

$$V^0(T,x,\omega) = \theta(x)$$

so that $EV^0(t,x,\omega) = \inf_{u \in \mathcal{A}} E[\theta(x(T;t,x,\omega)]$. The cost of perfect information known in the stochastic programming literature as EVPI (expected value of perfect information) [16],[19] is a measure of the effect of future randomness on the stochastic optimal control problem.

In order to obtain explicit equations and formulae for the cost functions, optimal controls, Lagrange multipliers and for the cost of information we will make smoothness and boundedness assumptions using a dynamic programming approach. Most of these assumptions are made to be able to prove the dynamic programming equation for the almost sure optimal control by which the anticipative control problem is solved. These assumptions ensure this equation has a solution (the value function) which can be represented in terms of stochastic flows. The same applies to the cost of perfect information. Some of these assumptions can be relaxed in particular cases as we will see in sections 2.4 and 3.3 . If we give up these computational aims we can weaken the assumptions by an approach based on applying pathwise the deterministic maximum principle (see Remark 4 after Proposition 2.1).

The outline of the report is as follows. In Section 1 we define the solution of a stochastic differential equation with anticipative controls in the drift by using the decomposition of the flow of a SDE written as the composition of the stochastic flow of the diffusion part (which does not involve controls) with the flow of a family of ordinary differential equations (ODE) parametrized by ω. Controls only appear in this random ODE and they can anticipate as stochastic integrals are not involved.In our paper [6] we used anticipative stochastic calculus and the result of [12] on the existence and uniqueness of solutions of drift anticipative SDEs in a suitable Sobolev space over the Wiener space. This approach requires Wiener smoothness and boundedness assumptions on the controls. We also present in this Section the stochastic characteristics method which allows the representation of solutions of SPDEs in terms of stochastic flows of SDEs. This representation will be very useful in finding Poisson bracket conditions for the existence of global solutions and for proving the various convergence results for SPDEs via the existing convergence results for SDEs. In Section 2 we prove the Lagrange multiplier theorem giving an explicit formula for $\lambda(t,x,\omega)$ in terms of the unique global solution of a linear backward HJB SPDE showing that the multiplier has all the properties required for it to act as an equality constraint for the nonanticipativity constraint for control processes. We consider as an example the nonanticipative LQG problem which is solved pathwise by determining first the Lagrange multiplier process. In Section 3 we solve the anticipative optimal control problem

$$\inf_{u \in \mathcal{A}} E[\theta(x(T)]$$

and we show that the value function of the pathwise problems by means of which it is solved is the unique global solution of a nonlinear backward Stratonovich SPDE. We impose Poisson bracket conditions for random conservation laws or for the invariance of the Lagrangian submanifold for the stochastic characteristic system of the nonlinear SPDE. Such conditions imply the existence of a global characteristic solution. A formula is obtained for the cost of perfect information and we prove that this is zero when the Lagrangian submanifold is invariant or there exists a time varying non random conservation law for the stochastic characteristic system of the HJBSPDE of anticipative optimal control. We consider as examples the anticipative LQG problem and a scalar nonlinear anticipative optimal control problem. We show that there exists indeed a random conservation law in the LQG case and we calculate the cost of perfect information. The Lagrangian submanifold is invariant in the nonlinear problem case and the cost of perfect information is zero.

1 : Anticipative controls, solutions of anticipative SDE and stochastic characteristics solutions for SPDE

Let $T>0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P, (w_t)_{0\le t\le T})$ denote the canonical d-fold Wiener space, i.e. $\Omega = C([0,T], R^d)$, $w_t(\omega) = \omega(t)$ is the coordinate process, $(\mathcal{F}_t^o)$ is the natural filtration of (w_t), P is Wiener measure, $\mathcal{F}$ is the P-completion of $\mathcal{F}_T^o$ and, for each $t \in [0, T]$, $\mathcal{F}_t$ is $\mathcal{F}_t^o$ completed with all P-null sets of $\mathcal{F}$. Thus (w_t) is a standard d-dimensional Brownian motion. Define $\tilde{\Omega}=[0,T]\times\Omega$, $\tilde{\mathcal{F}}=\mathcal{B}[0,T] \times \mathcal{F}$, $\tilde{P}$ = LebxP and let $\mathcal{P}$ be the σ-field of $\mathcal{F}_t$-predictable sets in $\tilde{\Omega}$. Fix an integer m and define $\mathcal{A}$ to be the set of functions $u : [0, T] \times \Omega \to \mathcal{U} \subset \mathbb{R}^m$ which are measurable with respect to the product σ-field $\mathcal{B}[0,T]\times\mathcal{F} = \tilde{\mathcal{F}}$ where $\mathcal{U}$ is compact. Define also $\mathcal{N}$ as the set of functions $u: [0, T] \times \Omega \to \mathcal{U} \subset \mathbb{R}^m$ which are measurable with respect to $\mathcal{P}$ and thus $\mathcal{F}_t$ adapted. Consider the nonlinear stochastic system

$$d x_t = \overline{f}(x_t, u_t)dt + g(x_t) \circ dw_t,\ x_0 = x \in \mathbb{R}^d \tag{1.1}$$

$$\overline{f}(x, u) = f(x, u) - \frac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(x) \ ;\ g_{i_x} = \frac{\partial g_i}{\partial x}$$

For a nonanticipative control $u_t \in \mathcal{N}$ the solution of (1.1) is defined via Stratonovich stochastic integral equation

$$x_t = x_0 + \int_0^t \overline{f}(x_\tau, u_\tau)d\tau + \int_0^t g(x_\tau) \circ dw_\tau$$

For anticipative controls $u_\tau \in \mathcal{A}$ we introduce the following definition assuming for the moment that g is C_b^2 and bounded; f is bounded and C^1 in (x, u).

Definition 1

For $u_t \in \mathcal{A}$ we call a solution of (1.1) the process x_t defined almost surely (i.e. for $\omega \in \Omega' \subset \Omega$, $P(\Omega')=1$) by

$$x_t(x) = \xi_t \circ \eta_t(x) \tag{1.2}$$

where

$$d\xi_t(y) = g(\xi_t(y)) \circ dw_t, \ \xi_0(y) = y \tag{1.3}$$

$$\frac{d\eta_t(x)}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t(x))\, \overline{f}(\xi_t \circ \eta_t(x), u_t); \tag{1.4}$$

$$\eta_0(x) = x$$

The solution (1.2) is defined almost surely because $\xi_t(y)$ is flow of diffeomorphisms a.s. so η_t is defined almost surely too.We see that this solution is well defined whether or not u_t is adapted because η_t is not defined by stochastic integrals and ξ_t does not depend on the control. This definition implies that we also defined, in a particular case, a generalized Stratonovich integral

$$\int_0^t g(x_\tau) \cdot dw_\tau$$

for x_τ the solution of (2) to be

$$\int_0^t g(x_\tau) \circ dw_\tau := x_t - x_0 - \int_0^t \overline{f}(x_\tau, u_\tau)\, d\tau$$

This "decomposition of the solution" of (1.1) - approach is used in [15] for defining solutions to SDE with anticipative drift , but a different generalized Stratonovich integral is considered there which requires additional smoothness assumptions. Using the anticipative Ito formula, (1.2) can be shown to be the solution of (1.1) in the sense of Ocone and Pardoux if additional smoothness is imposed on f,g and the anticipative controls have bounded Wiener space derivative $|D_s u(t,\omega)| \le M$. We pursued this anticipative stochastic calculus approach in [6]. We assume g is C_b^2 so that the solution of (1.3) exists for $\xi_0 = y$ and $\xi_t(y)$ is a C^1 flow of diffeomorphisms a.s. ;thus $\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(y)$ is well defined for all (t, y) a.s. according to Bismut [3, p.50]. Under these assumptions , (1.4) has a global solution (see Ocone and Pardoux [15]).

We present next the notion of stochastic characteristics (global) solution (developed in Kunita [12,13]) which will be used throughout for the stochastic partial differential equations involved in our approach. Consider the first order Stratonovich nonlinear SPDE for $x \in \mathbf{R}^d$, $t \in \mathbf{R}_+$ with initial condition

$$\begin{cases} dv(t,x) = F(t,x,\frac{\partial v}{\partial x})dt + \sum_{j=1}^{d} G_j(t,x,\frac{\partial v}{\partial x}) \circ dw_t^j \\ v(0,x) = \theta(x) \end{cases} \tag{1.5}$$

with $F(t,x,p)$ continuous in (t,x,p) and a $C^{m+1,\alpha}$ – function of (x,p) (i.e. m+1-times continuously differentiable in (x,p) with α-Hölder continuous $m+1^{th}$ partial derivatives, $\alpha > 0$), $G_j(t,x,p)$ $j=1,\dots,d$ continuous in (t,x,p), continuously differentiable in t and $C^{m+2,\alpha}$ – functions of (x,p) for $m \ge 3$ and θ is continuously differentiable. $w_t(\omega)$ is a standard d-dimensional Brownian motion as above.

A random field $v(t,x)$ defined for all $t \in \mathbf{R}_+$; $x \in \mathbf{R}$, will be said *global $C^{m,\alpha}$ – process* if for almost all $\omega \in \Omega$ $v(t,\cdot,\omega)$ is a $C^{m,\alpha}$ – function for all $t \in \mathbf{R}_+$ with continuous in (t,x) partial derivatives $\frac{\partial^k v(t,x,\omega)}{\partial x^k}$, $|k| \le m$ where $\frac{\partial^k}{\partial x} = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \dots \left(\frac{\partial}{\partial x_d}\right)^{k_d}$, $|k| = k_1 + \dots k_d$. A global $C^{m,\alpha}$ - process is a *global $C^{m,\alpha}$ - semimartingale* if $\frac{\partial^k v(\cdot,x,\omega)}{\partial x^k}$, $|k| \le m$ are semimartingales for each $x \in \mathbf{R}^d$.

Definition 2 A random field $v(t,x)$ is a ***global $C^{d,\alpha}$ solution*** of (1.5) if it is a global $C^{d,\alpha}$ – semimartingale and it satisfies for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}^d$

$$v(t,x) = \theta(x) + \int_0^t F\left(\tau,x,\frac{\partial v(\tau,x)}{\partial x}\right)d\tau + \sum_{j=1}^{d} \int_0^t G_j\left(\tau,x,\frac{\partial v}{\partial x}(\tau,x)\right) \circ dw_\tau^j \qquad \text{a.s.}$$

It is proved in [12,13] that the linear SPDE with initial condition $\theta(\cdot)$ of class $C^{l+1,\alpha}$, $2 \leq l \leq m$

$$\begin{cases} dv(t,x) = F(t,x)\frac{\partial v}{\partial x}(t,x)\,dt + \sum_{j=1}^{d} G_j(t,x)\frac{\partial v}{\partial x}(t,x) \circ dw_t^j \\ v(0,x) = \theta(x) \end{cases} \tag{1.6}$$

having F continuous in (t,x) and of class $C^{m+1,\alpha}$ and G_j continuous in (t,x), of class $C^{m+2,\alpha}$ $(m \geq 3)$ has a global (unique) $C^{l-1,\beta}$ solution with $\beta < \alpha$. Generally (1.5) has only a local solution v(t,x) up to a stopping time $t \leq T(x)$ which requires the definition of local random fields and local $C^{m,\alpha}$ – semimartingales [12,13]. This solution is represented as

$$v(t,x) = \nu_t \circ \phi_t^{-1}(t) \tag{1.7}$$

where $\varphi_t(x) = \tilde{\varphi}_t(x,\theta_x(x))$, $\chi_t(x) = \tilde{\chi}_t(x,\theta_x(x))$, $\nu_t(x) = \tilde{\nu}_t(x,\theta(x),\theta_x(x))$ are the flows of the *stochastics characteristics system* with general initial condition

$$d\tilde{\varphi}_t = -F_p(t,\tilde{\varphi}_t,\tilde{\chi}_t)dt + \sum_{j=1}^{d} G_{j,p}(t,\tilde{\varphi}_t,\tilde{\chi}_t) \circ dw_t^j \tag{1.8}$$

$$d\tilde{\chi}_t = F_x(t,\tilde{\varphi}_t,\tilde{\chi}_t)dt + \sum_{j=1}^{d} G_{j,x}(t,\tilde{\varphi}_t,\tilde{\chi}_t) \circ dw_t^j \tag{1.9}$$

$$d\tilde{\nu}_t = (F(t,\tilde{\varphi}_t,\tilde{\chi}_t) - F_p(t,\tilde{\varphi}_t,\tilde{\chi}_t)\,\tilde{\chi}_t)dt + \sum_{j=1}^{d} (G_j(t,\tilde{\varphi}_t,\tilde{\chi}_t) - G_{j,p}(t,\tilde{\varphi}_t,\tilde{\chi}_t)\tilde{\chi}_t) \circ dw_t^j \tag{1.10}$$

$$\tilde{\varphi}_0 = a,\quad \tilde{\chi}_0 = b\,,\quad \tilde{\nu}_0 = c$$

Here $F_p(t,x,p)$ denotes the vector of partial derivatives with respect to p. The reason for the solution (1.7) being only locally defined up to a stopping time $t \in T(x)$ is that $\varphi_t(x)$ is in general only a local flow of diffeomorphisms because of the "coupling" between (1.8) – (1.9).

We will be interested only in global solutions as these will characterize value functions of various optimal control problems . That is why we do not present in this introduction the local theory developed in [12,13]. We will be forced to find conditions ensuring (1.8) – (1.10) has certain (almost surely) invariant submanifolds "decoupling" from one another the stochastic characteristics and ensuring the existence of $\varphi_t^{-1}(x)$ for all $t \in \mathbf{R}_+$ a.s. thus turning (1.7) into a global solution. From this point of view the Poisson bracket conditions of §3.1 represent new results for the existence of global solutions for nonlinear SPDE's. The coefficients of our SPDE's will be assumed to have bounded derivatives (thus $\alpha=1$) and all our solutions will be global $C^{2,\beta}$ for $\beta < 1$. We will term them *(global) C^2 solutions* omitting β. Also, we will use backward SPDE's. Using Ito's backward formula [11] and replacing in (1.8) – (1.10) the forward Stratonovich integral by the backward Stratonovich integral, the characteristics representation of solution of a backward SPDE is again (1.7) but with backward characteristics.

2 : Families of deterministic control problems and nonanticipativity constraints via Lagrange multipliers

We shall show that the standard (nonanticipative) stochastic optimal control problem

$$\inf_{u \in \mathcal{N}} E[\theta(x(T))] \tag{2.1}$$

for (1.1) can be solved by allowing a larger class of controls $u \in \mathcal{A}$ (possibly anticipative), introducing nonanticipativity constraints in the cost via Lagrange multipliers and solving instead a family of deterministic optimal control problems parametrized by the sample functions of the Wiener process and of the Lagrange multipliers, $\lambda(t, x, \omega)$ with $\lambda : [0, T] \times \mathbf{R}^d \times \Omega \to \mathbf{R}^m$:

$$\inf_{u \in \mathcal{A}} [\theta(x_T(x, \omega)) + \int_0^T \lambda^T (\tau, x_\tau(x, \omega), \omega)\, u(\tau)\, d\tau] \tag{2.2}$$

with $x_t(x, \omega)$, the solution of (1.1) for anticipative controls as defined in Definition 1, $x_t(x, \omega) = \xi_t \circ \eta_t(x)$. As only $\eta_t(x)$ depends on u_t we get the equivalent family of problems for $\omega \in \Omega'$ with Ω' defined in Definition 1

$$\inf_{u \in \mathcal{A}} [\theta \circ \xi_T (\eta_T) + \int_0^T \tilde{\lambda}^T (\tau, \eta_\tau, \omega)\, u(\tau)\, d\tau] \tag{2.3}$$

$$\frac{d\eta_t}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t)\, \overline{f}(\xi_t \circ \eta_t, u_t),\ \eta_0 = x \tag{2.4}$$

where we dropped x from the notation $\eta_t(x)$ for simplicity and

$$\tilde{\lambda} (\tau, \eta, \omega) := \lambda(\tau, \xi_\tau(\eta), \omega) \tag{2.5}$$

By averaging (2.3) over the sample space for suitably chosen L^1 Lagrange multipliers one would then like to obtain the optimal cost (2.1). We will state and prove in this section a Lagrange multiplier theorem showing how to define $\hat{\lambda}(\tau, \eta, \omega)$ so that it will act as a nonanticipativity constraint in a family of deterministic (anticipative) optimal control problems.

We first make the following assumptions

(a) f is bounded in x and u and with continuous and bounded in x and u mixed partial derivatives up to order 5 (i.e. C_b^5); g is bounded and C_b^5; θ is C_b^4.

(b) For every $y \in \mathbb{R}^d$, $\sum_{i,j=1}^{d} (gg^T(x))_{ij}\, y_i y_j > 0$ for all $x \in \mathbb{R}^d$.

(c) The (nonanticipative) stochastic optimal control problem

$$
\text{(P)}\quad
\begin{aligned}
& dx_t = f(x_t, u_t)\,dt + g(x_t)\,dw \\
& x_0 = \bar{x} \\
& \inf_{u \in \mathcal{N}} E[\theta(x(T))]
\end{aligned}
\qquad (2.6)
$$

has a feedback solution $u^*(t, x)$ which is continuous and C^1 in t, C_b^4 in x. $u^*(t, x) \in \operatorname{int} \mathcal{U}$ for all $(t, x) \in [0,T] \times \mathbb{R}^d$ where $\mathcal{U}$ is the control value set assumed to be a compact subset of $\mathbb{R}^m$.

(d) The matrix with (i,j) element given in repeated indices summation convention notation

by $\dfrac{\partial[\theta \circ \xi_T \circ \psi_t^{-1}(\eta)]}{\partial \eta_k} \left(\left(\dfrac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\right)_{kl} \dfrac{\partial^2 f_l}{\partial u_i \partial u_j} (\xi_t(\eta), u)$ has characteristic values bounded below by some $\gamma > 0$ for all $(t, \eta, u) \in [0, T] \times \mathbb{R}^d \times \mathcal{U}$ a.s. where ψ_t^{-1} is the inverse flow of

$$\frac{d\psi_t(\eta)}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t(\eta))\, [(f(\xi_t(\psi_t(\eta)), u^*(t, \xi_t(\psi_t(\eta)))) - \tfrac{1}{2} \sum_{i=1}^{d} g_{ix} g_i($$

$$(\xi_t(\psi_t(\eta))) - \frac{\partial f}{\partial u}(\xi_t(\psi_t(\eta)), u^*(t, \xi_t(\psi_t(\eta)))) u^*(t, \xi_t(\psi_t(\eta)))]; \psi_T(\eta) = \eta \qquad (2.7)$$

Assumption (d) ,as will be seen in the next section ensures the strict convexity in u of $\frac{\partial W}{\partial \eta}(t,\eta)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta) f(\xi_t(\eta),u)$ which together with the particular choice of the Lagrange multiplier yields $u^*(t,x)$ as the unique minimizer of

$$\min_{u \in \mathcal{U}} \left\{\frac{\partial W}{\partial \eta}(t,\eta)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta) f(\xi_t(\eta),u) + \tilde{\lambda}^T(t,\eta,\omega)u\right\}$$

The characteristic representation is used to write the strict convexity condition in the form appearing in (d).

Assumptions (a, b, c) imply that the value function of the problem (2.6), $V^*(t, x)$, is the $C^{1,2}$ (i.e. C^1 in t, C^2 in x) unique solution of

$$\frac{\partial V^*(t, x)}{\partial t} + \min_{u \in \mathcal{U}} \{\frac{\partial V^*}{\partial x}(t, x) f(x, u)\} + \tfrac{1}{2}\mathrm{tr}\,(\frac{\partial^2 V^*}{\partial x^2} gg^T) = 0$$

$$V^*(T, x) = \theta(x) \qquad (2.7')$$

The assumption that f and g must be bounded can be relaxed only as far as allowing them to have "sublinear growth" [15] i.e. $\forall, \epsilon>0 \; \exists, K_\epsilon$ s.t. $| g(x) | \leq k_\epsilon (1+ | x |^{1-\epsilon}$. This is because of using the decomposition $x_t = \xi_t \circ \eta_t$ and because of needing to ensure that η_t (given by (2.4)) does not explode.

2.1 Main results

Our main Lagrange multiplier theorem is the following

Theorem 2.1

Consider the following family of deterministic optimal control problems indexed by $\omega \in \Omega' \subset \Omega$ $P(\Omega')=1$ (see Definition 1)

$$\dot{\eta}_t = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t)\,[f(\xi_t(\eta_t), u_t) - \tfrac{1}{2}\sum_{i=1}^{d} g_{i_x} g_i(\xi_t(\eta_t))],\ \eta_0 = x_0 \tag{2.8}$$

(P^ω)

$$\inf_{u \in \mathcal{M}} [\int_0^T \lambda^T(t, \eta_t, \omega)\, u(t)dt + \theta \circ \xi_T(\eta_T)]$$

where $\xi_t(\eta)$ is the solution of

$$d\xi_t = g(\xi_t) \circ dw_t \tag{2.9}$$

$$\xi_0 = \eta$$

Assume (a) - (d) and define

$$\lambda^T(t, \eta, \omega) := -\frac{\partial[\theta \circ \xi_T \circ \xi_t^{-1}(\eta)]}{\partial \eta} \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\, f_u\,(\xi_t(\eta), u^*(t, \xi_t(\eta))) \tag{2.10}$$

where $f_u = [\frac{\partial f_i}{\partial u_j}]$; $1 \leq i \leq d$, $1 \leq j \leq m$.

Then $u^*(t, \xi_t(\eta))$ is optimal for (P^ω) a.s. and so $u^*(t, x)$ is optimal for (2.2) and if we denote by $W(t, \eta)$ the value function of (P^ω) we have

$$W(t, \eta) = \theta \circ \xi_T \circ \xi_t^{-1}(\eta)$$

$$EW(t, \xi_t^{-1}(x)) = V^*(t, x)$$

$$E\, \tilde{\lambda}^T (t, \xi_t^{-1}(x)) = 0 \tag{2.11}$$

and $V(t, x) := W(t, \xi_t^{-1}(x))$ is the value function for the problem (1.2)-(1.3)-(2.2).

The basic idea in the proof of this theorem is to use first the deterministic dynamic programming approach pathwise for (2.3),(2.1) to obtain a random (robust) HJB PDE. The Lagrange multiplier will be chosen so that the minimizer in the HJB PDE is the same as the nonanticipative (feedback) optimal control $u^*(t,\xi_t(\eta))$ assumed to take values in the interior of $\mathcal{U}$. For this we need the strict convexity in u of the expression to be minimized in HJB PDE and $\tilde{\lambda}$ to be such that the derivative in u of this expression vanishes at $u^*(t,\xi_t(\eta))$ (interior minimum).The characteristics method will be used to ensure that there exists a $C^{1,2}$ solution of this pathwise HJB PDE so that using the verification theorem [8,p.87] $u^*(t,\xi_t(\eta))$ will turn out to be optimal for (2.2).To characterize the value function $W(t,\eta)$ in terms of x as solution of a HJB SPDE we will approximate the Wiener process by linear interpolation and we will show that HJB SPDE appears as limit of a sequence of random HJB PDE's.Averaging HJB SPDE we get the second order parabolic PDE of stochastic dynamic programming.This way our proof extends the Wong-Zakai type results [20] from differential equations to optimal control problems (dynamic programming equations)(see also the remark after the proof of Theorem 2.1 and [5] where the extension by continuity of the cost to C^0 paths was used). It is for this very important reason that we will prefer a rather lengthy approximation argument for the proof of Proposition 2.1 below. One can try to obtain the value function in terms of x by proving a generalization of Ito extended rule to calculate the differential of $W(t,\xi_t^{-1}(x))$ as the random field $W(t,\eta)$ is non adapted.This will directly show that the random HJB PDE is the robust equation associated to HJB SPDE via the transformation $x=\xi_t(\eta)$ given by the solution decomposition formula (1.2). We used the anticipative Ito rule of Ocone and Pardoux [15] which requires Wiener derivative growth estimates for the random field $W(t,\eta)$. This approach was developed in [6].

We start by proving the following result which is essential in the proof of Theorem 2.1

Proposition 2.1

Assume (a)-(d). The value function of the problems (P^ω) with $\tilde{\lambda}$ (t, η, ω) defined by (2.10) is the unique $C^{1,2}$ solution of the random PDE

$$\frac{\partial W}{\partial t}(t, \eta) + \frac{\partial W}{\partial \eta}(t, \eta)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\,[f(\xi_t(\eta), u^*(t, \xi_t(\eta)))$$

$$-\frac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(\xi_t(\eta)) - \frac{\partial f}{\partial u}(\xi_t(\eta), u^*(t, \xi_t(\eta))\, u^*(t, \xi_t(\eta)))] = 0 \qquad (2.12)$$

$$W(T, \eta) = \theta \circ \xi_T(\eta)$$

where $\xi_t(\eta)$ is the unique nonexploding solution of (2.9). $V(t, x) := W(t, \xi_t^{-1}(x))$ is the unique C^2 solution of the backward stochastic partial differential equation (SPDE)

$$dV(t, x) + \frac{\partial V}{\partial x}(t, x)\,[f(x, u^*(t, x)) - \frac{\partial f}{\partial u}(x, u^*(t, x))\, u^*(t, x)$$

$$-\frac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(x)]\, dt + \frac{\partial V}{\partial x}(t, x)\, g(x) \circ dw_t = 0$$

$$V(T, x) = \theta(x) \qquad (2.13)$$

or in Ito backward form

$$dV(t, x) + \{\frac{\partial V}{\partial x}(t, x)\,[f(x, u^*(t, x)) - \frac{\partial f}{\partial u}(x, u^*(t, x))\, u^*(t, x)\,]$$
$$+ \frac{1}{2}\,\mathrm{tr}\left(\frac{\partial^2 V}{\partial x^2}(t, x)\, gg^T(x)\right)\}\, dt + \frac{\partial V}{\partial x}(t, x)\, g(x)\ \hat{d}w_t = 0$$
$$V(T, x) = \theta(x) \qquad (2.14)$$

Remarks

1. (2.12) is thus the "robust equation" associated with the SPDE (2.13) (see Cannarsa and Vespri [4])

2. "$\hat{d}$" and "$\circ\,\hat{d}$" denote the differentials corresponding respectively to Ito's and Stratonovich's backward integrals (see Kunita [14])

3. A stochastic Hamilton-Jacobi partial differential equation was considered in Bismut [3, p. 323] in the context of random mechanics

4. A different approach can be pursued based on pathwise application of the deterministic maximum principle. The Lagrange multiplier can then be defined in terms of a random adjoint process. As such an approach will not involve SPDE's and thus we will not make use of stochastic characteristics the smoothness assumptions can be considerably weakened.

2.2 Proof of the main results

Two lemmas are required in order to prove Proposition 2.1 and thus Theorem 2.1.

Lemma 2.1

Consider the sequence of random PDE's

$$\frac{\partial W^n}{\partial t}(t, \eta) + \frac{\partial W^n}{\partial \eta}(t, \eta)\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}(\eta)\,[f\,(\xi_t^n(\eta), u^*(t, \xi_t^n(\eta))$$

$$-\frac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(\xi_t^n(\eta)) - \frac{\partial f}{\partial u}(\xi_t^n(\eta), u^*(t, \xi_t^n(\eta)))\, u^*(t, \xi_t^n(\eta))] = 0$$

$$W^n(T, \eta) = \theta \circ \xi_T^n(\eta) \tag{2.15}$$

with

$$\frac{d\xi_t^n}{dt}(\eta) = g\,(\xi_t^n(\eta))\,\dot{v}_t^n$$

$$\xi_0^n(\eta) = \eta \tag{2.16}$$

where $v^n(t, \omega)$ is the piecewise linear approximation ("linear interpolation") of the d-dimensional Wiener process $w(t, \omega)$:

$$v^n(t, \omega) = w(t_k) + 2^n(t - t_k)\,(w(t_{k+1}) - w(t_k));\ t \in [t_k, t_{k+1})$$

where $t_k = k/2^n$; $k = 0, 1, \ldots, [T/2^n]$ so that

$$\dot{v}^n(t, \omega) = 2^n\,(w(t_{k+1}) - w(t_k)) \tag{2.17}$$

for $t \in [t_k, t_{k+1}]$.

Under the assumptions (a, c) the sequences of random PDE's (2.15) have unique $C^{1,2}$ solution for each

$n \in \mathbb{N}$ and $V^n(t,x) := W^n(t, (\xi_t^n)^{-1}(x))$ satisfies :

$$\frac{\partial V^n}{\partial t}(t,x) + \frac{\partial V^n}{\partial x}(t,x)\,[f(x,u^*(t,x)) - \tfrac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(x) - \frac{\partial f}{\partial u}(x,u^*(t,x))$$

$$\times\; u^*(t,x)] + \frac{\partial V^n}{\partial x}\, g(x)\, \dot{v}_t^n = 0 \tag{2.18}$$

$$V^n(T,x) = \theta(x)\;.$$

Proof

For the 'piecewise' linear approximations v_t^n considered above we have

$$\frac{\partial V^n}{\partial t}(t,x) = \frac{\partial W^n(t,(\xi_t^n)^{-1}(x))}{\partial t} + \frac{\partial W^n}{\partial \eta}(t,(\xi_t^n)^{-1}(x))\,\frac{d(\xi_t^n)^{-1}(x)}{dt}$$

$$= -\frac{\partial W^n}{\partial \eta}(t,(\xi_t^n)^{-1}(x))\left(\frac{\partial(\xi_t^n)^{-1}}{\partial x}\right)^{-1}((\xi_t^n)^{-1}(x))\,[f(x,u^*(t,x)) - \tfrac{1}{2}\sum_{i=1}^{d} g_{i_x}\, g_i(x)$$

$$- \frac{\partial f}{\partial u}(x,u^*(t,x))\,u^*(t,x)] - \frac{\partial W^n}{\partial \eta}(t,(\xi_t^n)^{-1}(x))\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}((\xi_t^n)^{-1}(x))\, g(x)\, \dot{v}_t^n$$

because

$$\frac{d(\xi_t^n)^{-1}(x)}{dt} = -\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}((\xi_t^n)^{-1}(x))\, g(x)\, \dot{v}_t^n\;.$$

Using now

$$\frac{\partial V^n}{\partial x}(t,x) = \frac{\partial W^n}{\partial \eta}(t,(\xi_t^n)^{-1}(x))\,\frac{\partial(\xi_t^n)^{-1}(x)}{\partial x}$$

$\left(\frac{\partial(\xi_t^n)^{-1}}{\partial x}(x)\right)^{-1}\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}((\xi_t^n)^{-1}(x)) = I_n$ (due to (a), $\xi_t^n(x)$ is a.s. a flow of diffeomorphisms and $\xi_t^n \circ (\xi_t^n)^{-1}(x) = x$)

we get the required sequence of random PDE's for $V^n(t, x)$. To see that indeed under the assumptions made, $W^n(t, \eta)$ and $V^n(t, x)$ exist and are a.s. $C^{1,2}$, we consider the corresponding characteristic equations (obtained by replacing $\xi_t(\eta)\left(\frac{\partial \xi_t}{\partial t}\right)^{-1}(\eta)$ in (2.7) by $\xi_t^n(\eta)\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}(\eta)$ to get $\iota_t^n(\eta)$) and representations of the solutions for (2.15) and (2.18). Reasoning similarly to [14, Lemma 6.2(4)] for the integral form of random ordinary differential equations we write forward equations for the inverse of the flow of characteristics for each fixed $t \in [0, T]$):

$$(\iota_t^n)^{-1}(\eta) = (\iota_{tT}^n)^{-1}(\eta) : \iota_t^n \circ (\iota_t^n)^{-1}(\eta) = \eta$$

where

$$\frac{d(\iota_{ts}^n)^{-1}(\eta)}{ds} = \left(\frac{\partial \xi_s^n}{\partial x}\right)^{-1} ((\iota_{ts}^n)^{-1}(\eta)) \, [f(\xi_s^n((\iota_{ts}^n)^{-1}(\eta)), u^*(s, \xi_s^n((\iota_{ts}^n)^{-1}(\eta))))$$

$$- \frac{1}{2} \sum_{i=1}^{d} g_{i_x} g_i(\xi_s^n((\iota_{ts}^n)^{-1}(\eta))) - \frac{\partial f}{\partial u}(\xi_s^n((\iota_{ts}^n)^{-1}(\eta)), u^*(s, \xi_s^n((\iota_{ts}^n)^{-1}(\eta))))$$

$$\times \; u^*(s, \xi_s^n((\iota_{ts}^n)^{-1}(\eta)))] := a^n(s, (\iota_{ts}^n)^{-1}(\eta), \omega) \tag{2.19}$$

$$(\iota_{tT}^n)^{-1}(\eta) = \eta : t \le s \le T$$

and respectively

$$(\zeta_t^n)^{-1}(x) = (\zeta_{tT}^n)^{-1}(x) : \zeta_t^n \circ (\zeta_t^n)^{-1}(x) = x$$

where

$$\frac{d(\zeta_{ts}^n)^{-1}(x)}{ds} = f((\zeta_{ts}^n)^{-1}(x), u^*(s, (\zeta_{ts}^n)^{-1}(x))) - \frac{1}{2} \sum_{i=1}^{d} g_{i_x} g_i((\zeta_{ts}^n)^{-1}(x))$$

$$- \frac{\partial f}{\partial u}((\zeta_{ts}^n)^{-1}(x), u^*(s, (\zeta_{ts}^n)^{-1}(x)) \, u^*(s, (\zeta_{ts}^n)^{-1}(x)) + g((\zeta_{ts}^n)^{-1}(x)) \, \dot{v}_s^n$$

$$(\zeta_{tt}^n)^{-1}(x) = x : t \le s \le T \tag{2.20}$$

For each t, n, for almost all $\omega \in \Omega$ (2.15) and (2.20) have unique non-exploding solutions on $[t, T]$ under (a, c). It is clear that for bounded f which is C_b^2 in (x, u) and bounded g which is C_b^2, (2.20) has for every $n \in \mathbf{N}$, for almost all $\omega \in \Omega$ and $x \in \mathbf{R}^d$ a unique solution on $[t, T]$ [3, p. 38]. To see that

(2.19) has a non-exploding solution on $[t, T]$ we have to show that $a^n(s, x, \omega)$ satisfies a linear growth condition for each n a.s. We know from [3, p.39] and [15, p. 57] that for every t,T, $\beta > 0$

$$|\left(\frac{\partial \xi_s^n}{\partial x}\right)^{-1}(x)| \leq \ell_\beta^n(\omega)\,(1 + |x|^2)^\beta \;;\; s \in [t, T] \tag{2.21}$$

for some family of random variables $\{\ell_\beta^n(s)\}$ $n \in \mathbf{N}$ which is uniformly bounded in L_p for all $p \geq 1$ i.e.

$$\sup_{n \in \mathbf{N}} E\,|\,\ell_\beta^n\,|^p < \infty \tag{2.22}$$

and such that for every $\beta>0$ and $\alpha \geq 0$ there exist constants $C_{\alpha,\beta}$ for which

$$P(\ell_\beta^n > \ell) \leq \frac{C_{\alpha,\beta}}{\ell^\alpha} \tag{2.23}$$

for any $\ell>0$. In fact in [15] it is shown that $\left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(x)$ satisfies (2.21) using $E \sup_{s \leq T} |\,\xi_s(x)\,|^p \leq C_p(1 + |\,x\,|^2)^{p/2}$ for $p \geq 2$ and Sobolev's inequality. Using the fact that

$$\xi_s^n(x) - x = I_n(s, \omega) \text{ where } E\,[\sup_{s \leq T} |\,I_n(s, \omega)\,|^p] < \infty$$

from Ikeda and Watanabe [10, p. 506] one shows that also $E \sup_{s \leq T} |\,\xi_s^n(x)\,|^q \leq C_q^n(1 + |x|^2)^{q/2}$ for $q \geq 2$. Here c_q, c_q^n are constants for each fixed $q \geq 2$, $n \in \mathbf{N}$.

Due to the assumptions we made and due to (2.21) in which we take $\beta = \frac{1}{2}$

$$\begin{aligned} |\,F^n(s, x, \omega)\,| &:= |\,f(\xi_s^n(x), u^*(s, \xi_s^n(x))) - \tfrac{1}{2}\sum_{i=1}^d g_{i_x} g_i(\xi_s^n(x)) \\ &- \frac{\partial f}{\partial u}(\xi_s^n(x), u^*(s, \xi_s^n(x)))\, u^*(s, \xi_s^n(x))\,| \leq M \end{aligned} \tag{2.24}$$

and

$$|\,a^n(s, x, \omega)\,| \leq |\left(\frac{\partial \xi_s^n}{\partial x}\right)^{-1}(x)|\,|\,F^n(s, x, \omega)\,| \leq \bar{\ell}^n(\omega)\,(1 + |\,x\,|) \text{ a.s.} \tag{2.25}$$

for a uniformly L_p bounded family of random variables $\{\bar{\ell}^n(\omega)\}$ $n \in \mathbf{N}$ for any $p \geq 1$. The a.s. linear growth of $a^n(s, x, \omega)$ (2.25) ensures that the solution of (2.19) does not explode.

Thus we have the representation of the solution of (2.15) and (2.18) :

$$W^n(t, \eta) = \theta \circ \xi^n_T \circ (\upsilon^n_t)^{-1}(\eta) = \theta \circ \xi^n_T \circ (\upsilon^n_{tT})^{-1}(\eta) \tag{2.26}$$

$$V^n(t, x) = \theta \circ (\zeta^n_{tT})^{-1}(x) \tag{2.27}$$

The uniqueness of the $C^{1,2}$ solution for a.a.ω. of (2.15) follows from the uniqueness of the solution provided by the characteristics method and the uniqueness of the solutions of (2.19) and (2.20).

Lemma 2.2

Under the assumptions (a, c) for the linear interpolation approximation $v^n(t, \omega)$ as in the statement of Lemma 2.1 we have for each $t \in [0, T]$

$$V^n(t, x) \xrightarrow{P} V(t, x) \tag{2.28}$$

$$W^n(t, \eta) \xrightarrow{P} W(t, \eta) \tag{2.29}$$

$$W^n(t, (\xi^n_t)^{-1}(x)) \xrightarrow{P} W(t, \xi^{-1}_t(x)) \tag{2.30}$$

uniformly in x on compact subsets of $\mathbf{R}^d$.Here $V(t, x)$, $W(t, \eta)$ are the unique C^2 and respectively $C^{1,2}$ solutions of (2.1) and respectively (2.12). "$\xrightarrow{P}$" denotes here convergence in probability as $n \to \infty$.

Proof

For f and g satisfying (a, c) we get for each $t \leq T$

$$(\zeta^n_{tT})^{-1}(x) \xrightarrow{P} \zeta^{-1}_{tT}(x)) \tag{2.31}$$

where $\zeta^{-1}_{ts}(x)$ satisfies

$$d_s\, \zeta_{ts}^{-1}(x) = [f(\zeta_{ts}^{-1}(x), u^*(s, \zeta_{ts}^{-1}(x))) - \tfrac{1}{2}\sum_{i=1}^{d} g_{i_x} g_i(\zeta_{ts}^{-1}(x))$$

$$- \frac{\partial f}{\partial u}(\zeta_{ts}^{-1}(x), u^*(s, \zeta_{ts}^{-1}(x)))\, u^*(s, \zeta_{ts}^{-1}(x))]\, ds + g(\zeta_{ts}^{-1}(x)) \circ dw_s \tag{2.32}$$

$$\zeta_{tt}^{-1}(x) = x;\ t \le s \le T$$

This follows by applying the convergence result for the inverse of flows of SDE's with bounded coefficients [3, p.66 and p.39].

To prove (2.28) we use now the representation (2.27) of the solution of (2.18) given by the characteristics method. Assumptions (a, c) ensure the existence and uniqueness of $V^n(t, x)$ for each $n \in \mathbf{N}$ and due to (2.31) and (2.27) we have for each $t \in [0, T]$

$$V^n(t, x) \xrightarrow{P} \theta \circ \zeta_{tT}^{-1}(x) \tag{2.33}$$

uniformly in x on compact subsets of $\mathbf{R}^d$. But under the assumptions made we know from Kunita [12, Section VI] that the unique C^2 solution of (14) is $V(t, x) = \theta \circ \zeta_t^{-1}(x) = \theta \circ \zeta_{tT}^{-1}(x)$ where $\zeta_t(x)$ satisfies

$$d\,\zeta_t(x) = [f(\zeta_t(x), u^*(t, \zeta_t(x))) - \tfrac{1}{2}\sum_{i=1}^{d} g_{i_x} g_i(\zeta_t(x)) - \frac{\partial f}{\partial u}(\zeta_t(x),$$

$$u^*(t, \zeta_t(x)))\, u^*(t, \zeta_t(x))]\, dt + g(\zeta_t(x)) \circ dw_t \tag{2.34}$$

$$\zeta_T(x) = x\ .$$

and $\zeta_t^{-1}(x)$ is the unique solution of (2.32) for $s = T$ (Kunita [14]) so that (2.28) follows. The flow of (2.34) was denoted $\zeta_{tT}(x)$ as it has terminal condition at T.To prove (2.29), (2.30) we use the representation (2.26) given again by the characteristics method for PDE. We will prove first that for each $t \in [0, T]$

$$(\iota_{tT}^{n})^{-1}(\eta) \xrightarrow{P} \iota_{tT}^{-1}(\eta) \tag{2.35}$$

uniformly in η on compact subsets of $\mathbf{R}^d$. (2.35) will then be used in proving that the following convergences are valid uniformly on compact sets of $\mathbf{R}^d$, for each $t \in [0, T]$:

$$W^n(t, (\xi_t^n)^{-1}(x)) = \theta \circ \xi_T^n \circ (\iota_{tT}^n)^{-1} \circ (\xi_{0t}^n)^{-1}(x) \xrightarrow{P} W(t, \xi_{0t}^{-1}(x)) = \theta \circ \xi_T \circ \psi_{tT}^{-1} \circ \xi_{0t}^{-1}(x) \tag{2.36}$$

and

$$W^n(t, \eta) = \theta \circ \xi_T^n \circ (\iota_{tT}^n)^{-1}(\eta) \xrightarrow{P} W(t, \eta) = \theta \circ \xi_T \circ \psi_{tT}^{-1}(\eta) \tag{2.36'}$$

where

$$d\, \xi_{0s}^{-1}(x) = -\, g(\xi_{0s}^{-1}(x)) \circ dw_s^t \tag{2.37}$$

$$\xi_{00}^{-1}(x) = x : 0 \le s \le t$$

$$w_s^t = w_t - w_{t-s}$$

$$\xi_t^{-1}(x) = \xi_{0t}^{-1}(x)$$

$$\frac{d(\xi_{0s}^n)^{-1}(x)}{ds} = -\, g((\xi_{0s}^n)^{-1}(x)) \frac{d\iota_s^{t,n}}{ds} \tag{2.38}$$

$$(\xi_{00}^n)^{-1}(x) = x : 0 \le s \le t$$

$$\iota_s^{t,n} = v_t^n - v_{t-s}^n$$

$$(\xi_t^n)^{-1}(x) = (\xi_{0t}^n)^{-1}(x)$$

and

$$\frac{d\iota_{ts}^{-1}(\eta)}{ds} = \left(\frac{\partial \xi_s}{\partial x}\right)^{-1} (\psi_{ts}^{-1}(\eta))\, [f(\xi_s(\iota_{ts}^{-1}(\eta)),\, u^*(s, \xi_s(\psi_{ts}^{-1}(\eta))))$$

$$- \frac{1}{2} \sum_{i=1}^{d} g_{i_x} g_i(\xi_s(\psi_{ts}^{-1}(\eta))) - \frac{\partial f}{\partial u}(\xi_s(\psi_{ts}^{-1}(\eta)), u^*(s, \xi_s(\psi_{ts}^{-1}(\eta)))) \tag{2.39}$$

$$\times \; u^*(s, \xi_s(\psi_{ts}^{-1}(\eta)))] := a(s, \psi_{ts}^{-1}(\eta), \omega)$$

$$\psi_{tt}^{-1}(\eta) = \eta \; ; \; t \leq s \leq T$$

In what follows t is a fixed value of [0, T]. The systems of random differential equations (2.19) and (2.39) have unique nonexploding solutions $(\psi_{st}^{n})^{-1}$, ψ_{st}^{-1} a.s. due to the a.s. linear growth condition for $\beta=\frac{1}{2}$ (2.21) and to the a.s. locally Lipschitz property of $a^n(s, x, \omega)$, $a(s, x, \omega)$. The latter follows from (a, c) which imply that

$$\left(\frac{\partial \xi_s^n}{\partial x}\right)^{-1}(x), \left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(x),$$

are C^1 a.s. [3, p.50]. Take now K a compact in $\mathbf{R}^d$ and define for R>0

$$\tau_R^K = \inf\{s \in [t, T] \mid \sup_{\eta \in K} |\psi_{ts}^{-1}(\eta)| \geq R-1\} \wedge T$$

$$\tau_R^{K,n} = \inf\{s \in [t, T] \mid \sup_{\eta \in K} |(\psi_{ts}^{n})^{-1}(\eta)| \geq R\} \wedge T$$

Now for $0 \leq s \leq \tau_R^k \wedge \tau_R^{k,n}$ we have

$$A_s(\eta, \omega) := |(\psi_{ts}^{n})^{-1}(\eta) - \psi_{ts}^{-1}(\eta)| \leq \int_t^s |a^n(\tau, (\psi_{t\tau}^{n})^{-1}(\eta), \omega) - a(\tau, (\psi_{t\tau}^{n})^{-1}(\eta), \omega)|$$

$$\times \; d\tau + \int_t^s |a(\tau, (\psi_{t\tau}^{n})^{-1}(\eta), \omega) - a(\tau, \psi_{t\tau}^{-1}(\eta), \omega)| \, d\tau$$

$$\leq \int_t^s \sup_{\substack{t \leq \tau \leq \tau_R^k \wedge \tau_R^{k,n} \\ \eta \in K}} |a^n(\tau, (\psi_{t\tau}^{n})^{-1}(\eta), \omega) - a(\tau, (\psi_{t\tau}^{n})^{-1}(\eta), \omega)| \, d\tau$$

$$+ \int_t^s \sup_{\substack{t \le \tau \le \tau_R^k \wedge \tau_R^{k,n} \\ \eta \in K}} \left| \frac{\partial a}{\partial x}\left(\tau, \rho_{t\tau}^n(\omega)(\psi_{t\tau}^n)^{-1}(\eta) + (1 - \rho_{t\tau}^n(\omega))(\psi_{t\tau}^n)^{-1}(\eta), \omega\right) \right|$$

$$\times \left|(\psi_{t\tau}^n)^{-1}(\eta) - \psi_{t\tau}^{-1}(\eta)\right| d\tau \le T \sup_{\substack{t \le \tau \le T \\ |x| \le R}} |a^n(\tau, x, \omega) - a(\tau, x, \omega)|$$

$$+ \int_t^s \sup_{\substack{t \le \tau \le T \\ |y| \le 2R}} \left| \frac{\partial a}{\partial x}(\tau, y, \omega) \right| . A_\tau(\eta, \omega)\, d\tau$$

where $\rho_{t\tau}^n(\omega) \in [0, 1]$ a.s. for every $n \in \mathbf{N}$ are given by the mean value theorem applied a.s. for each n. Using the estimates of Ocone and Pardoux [15, p.57] we get the following estimate

$$\sup_{\substack{t \le \tau \le T \\ |y| \le 2R}} \left| \frac{\partial a}{\partial x}(\tau, y, \omega) \right| \le \sup_{\substack{0 \le \tau \le T \\ |y| \le 2R}} \left| \frac{\partial}{\partial x}\left(\left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\right) \right| |F| + \sup_{\substack{0 \le \tau \le T \\ |y| \le 2R}} \left| \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x) \right|$$

$$\times |F_x + F_u\, u_x^*| \sup_{\substack{0 \le \tau \le T \\ |y| \le 2R}} \left| \left(\frac{\partial \xi_\tau}{\partial x}\right)(x) \right| \le M\left(L(\omega)(1 + 4R^2) + (L(\omega)(1 + 4R^2))^2\right)$$

$$:= \ell(\omega)(1 + 4R^2)^2$$

where $L(\omega)$, $\ell(\omega)$ are L^P bounded r.v.'s . We use again the notation $a(\tau, x, \omega) = \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\, F(\xi_\tau(x),$ $u^*(\tau, \xi_\tau(x)))$. Due to (a, c), F is bounded and so is $F_x + F_u\, u_x^*$ where F_x, F_u denote Jacobians with respect to the first and respectively second variable. M denoting their common bound. u_x^* is the Jacobian of $u^*(t,x)$ with respect to the second variable. Applying now Gronwall's inequality a.s. we obtain

$$A_s(\eta, \omega) \le \exp\left(\ell(\omega)(1 + 4R^2)^2 T\right) . T \sup_{\substack{0 \le \tau \le T \\ |x| \le R}} |a^n(\tau, x, \omega) - a(\tau, x, \omega)| \quad \forall$$

$$s \le \tau_R^K \wedge \tau_R^{K,n}$$

We show next that

$$a^n(\tau, x, \omega) \xrightarrow{P} a(\tau, x, \omega)$$

uniformly in (τ, x) on compact subsets of $[0, T] \times \mathbb{R}^d$. We have by the mean value theorem for F

$$|a^n(\tau, x, \omega) - a(\tau, x, \omega)| \le \left|\left(\frac{\partial \xi^n_\tau}{\partial x}\right)^{-1}(x) - \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\right| |F| + \left|\left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\right|$$

$$\times |F_x + F_u u^*_x| \, |\xi^n_\tau(x) - \xi_\tau(x)| \le \sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} \left|\left(\frac{\partial \xi^n_\tau}{\partial x}\right)^{-1}(x) - \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\right| M + M\, L(\omega)$$

$$\times (1 + R^2) \sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} |\xi^n_\tau(x) - \xi_\tau(x)|$$

Using now the uniform convergences in probability on compacts

$$\xi^n_\tau(x) \xrightarrow{P} \xi_\tau(x)$$

$$\left(\frac{\partial \xi^n_\tau}{\partial x}\right)^{-1}(x) \xrightarrow{P} \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)$$

which hold under (a, c) [3, p.39], [10, p.516] we get $\forall$, $R>0$, ϵ, δ $\exists$, $N^R_{\epsilon,\delta}$ such that $\forall\, n > N^R_{\epsilon,\delta}$

$$P\Big(\sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} |a^n(\tau, x, \omega) - a(\tau, x, \omega)| < \delta\Big) \ge P((L(\omega) < L_\epsilon)$$

$$\& \Big(\sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} |\xi^n_\tau(x) - \xi_\tau(x)| < \frac{\delta}{2ML(1+R^2)}\Big) \,\&\, \Big(\sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} \left|\left(\frac{\partial \xi^n_\tau}{\partial x}\right)^{-1}(x)\right.$$

$$\left. - \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(x)\right| < \frac{\delta}{2M}\Big) \ge 1 - \frac{C}{L_\epsilon} + 1 - \frac{\epsilon}{3} + 1 - \frac{\epsilon}{3} - 2 = 1 - \epsilon \text{ for } L_\epsilon = \frac{3C}{\epsilon}$$

This implies that for a fixed compact $K \subset \mathbb{R}^d$ $\forall$, $R>0$, $\epsilon>0$, $\delta>0$ $\exists$, $N^{R,K}_{\epsilon,\delta}$ such that $\forall\, n > N^R_{\epsilon,\delta}$

$$P(\Lambda_s(\eta, \omega) < \delta \text{ for } t \le s \le \tau^k_R \wedge \tau^{k,n}_R \text{ and } \eta \in K) \ge P((\ell(\omega) < L_\epsilon)$$

$$\& \left(\sup_{\substack{0 \le \tau \le T \\ |x| \le 2R}} |a^n(\tau, x, \omega) - a(\tau, x, \omega)| < \delta \left(T\exp\left(L_\epsilon(1 + 4R^2)^2 T\right)\right)^{-1} \right) \ge 1 - \frac{C''}{L'_\epsilon} + 1$$

$$- \frac{\epsilon}{6} - 1 = 1 - \frac{\epsilon}{3} \text{ for } L'_\epsilon = \frac{6C''}{\epsilon}$$

so that using the same urgument as in Shreve and Karatzas [11, p. 298]

$$P(\tau_R^k \le \tau_R^{k,n}) \ge P\left(\sup_{\eta \in K} |(\psi_{ts}^n)^{-1}(\eta)| < \delta + \sup_{\eta \in K} |\psi_{ts}^{-1}(\eta)| \text{ for}\right.$$

$$\left. t \le s \le \tau_R^K \wedge \tau_R^{K,n}\right) \ge 1 - \epsilon/3$$

Finally, because

$$\lim_{R \to \infty} P\left(\sup_{\eta \in K} |\psi_{ts}^{-1}(\eta)| < R-1 \ \forall, s \in [t, T]\right) = 1$$

and so

$$\lim_{R \to \infty} P(\tau_R^K = T) = 1$$

(i.e. $\forall, \epsilon > 0 \ \exists, R_\epsilon^K$ such that $\forall, R \ge R_\epsilon^K$ $P(\tau_{R_\epsilon^k}^K = T) \ge 1 - \frac{\epsilon}{3}$) we obtain for any $t \in [0, T]$ and for any compact $K \subset \mathbb{R}^d$ $\forall, \epsilon, \delta > 0 \ \exists, N_{\epsilon,\delta}^{K, R_\epsilon^K} := \overline{N}_{\epsilon,\delta}^K$ such that $\forall, n \ge \overline{N}_{\epsilon,\delta}^K$

$$P\left(\sup_{\substack{t \le s \le T \\ \eta \in K}} A_s(\eta, \omega) < \delta\right) \ge P(A_s(\eta, \omega) < \delta \text{ for } t \le s \le T, \eta \in K)$$

$$\ge P((A_s(\eta, \omega) < \delta \text{ for } t \le s \le \tau_R^K \wedge \tau_R^{K,n}, \eta \in K) \& (\tau_R^K = T) \& (\tau_R^K \le \tau_R^{K,n}))$$

$$\ge 1 - \frac{\epsilon}{3} + 1 - \frac{\epsilon}{3} + 1 - \frac{\epsilon}{3} - 2 = 1 - \epsilon$$

and thus we managed to prove in particular for $s = T$ (2.35). We can now prove (2.36′).

For $t \le s \le \tau_R^K \wedge \tau_R^{K,n}$ we have

$$B_s(\eta,\omega) := |\theta \circ \xi^n_T \circ (\psi^n_{ts})^{-1}(\eta) - \theta \circ \xi_T \circ \psi^{-1}_{ts}(\eta)| \leq |\theta_x|\,|\xi^n_T \circ (\psi^n_{ts})^{-1}(\eta)$$

$$- \xi_T \circ \psi^{-1}_{ts}(\eta)| \leq M(|\xi^n_T \circ (\psi^n_{ts})^{-1}(\eta) - \xi_T \circ (\psi^n_{ts})^{-1}(\eta)| + |\xi_T \circ (\psi^n_{ts})^{-1}(\eta)$$

$$- \xi_T \circ \psi^{-1}_{ts}(\eta)|) \leq M|\xi^n_T \circ (\psi^n_{ts})^{-1}(\eta) - \xi_T \circ (\psi^n_{ts})^{-1}(\eta)| + M\left|\frac{\partial \xi_T}{\partial x}\right|(\gamma^n_{ts}$$

$$\times (\psi^n_{ts})^{-1}(\eta) + (1-\gamma^n_{ts})\psi^{-1}_{ts}(\eta))\,|(\psi^n_{ts})^{-1}(\eta) - \psi^{-1}_{ts}(\eta)| \leq M \sup_{|x| \leq R} |\xi^n_T(x)$$

$$- \xi_T(x)| + ML(\omega)(1+4R^2) \sup_{\substack{t \leq s \leq T \\ \eta \in \overline{K}}} |(\psi^n_{ts})^{-1}(\eta) - \psi^{-1}_{ts}(\eta)|$$

For a compact $K \subset \mathbb{R}^d$, $\forall\, \epsilon, \delta > 0\ \exists\, R^K_\epsilon$ and $\bar{L}_\epsilon = \frac{6\overline{C}}{\epsilon}$ and $\overline{\overline{N}}^{K,\,R^K_\epsilon}_{\epsilon,\delta} := \tilde{N}^K_{\epsilon,\delta}$ such that $\forall\, n \geq N^K_{\epsilon,\delta}$:

$$P(B_s(\eta,\omega) < \delta \text{ for } t \leq s \leq \tau^K_R \wedge \tau^{K,n}_R,\ \eta \in K) \geq P((\sup_{|x| \leq R^k_\epsilon} |\xi^n_T(x) - \xi_T(x)|$$

$$< \frac{\delta}{2M}) \,\&\, (L(\omega) < L) \,\&\, (\sup_{\substack{t \leq s \leq T \\ \eta \in \overline{K}}} |(\psi^n_{ts})^{-1}(\eta) - \psi^{-1}_{ts}(\eta)| < \frac{\delta}{2ML(1+4R^2)}))$$

$$\geq 1 - \frac{2\epsilon}{6} + 1 - \frac{\epsilon}{6} + 1 - \frac{\epsilon}{6} - 2 = 1 - \frac{2\epsilon}{3}$$

Hence we obtain

$$P(\sup_{\substack{t \leq s \leq T \\ \eta \in \overline{K}}} B_s(\eta,\omega) < \delta) \geq P((B_s(\eta,\omega) < \delta;\ t \leq s \leq \tau^K_R \wedge \tau^{K,n}_R,\ \eta \in K) \text{ and}$$

$$\text{and } (\tau^K_R = T) \text{ and } (\tau^K_R \leq \tau^{K,n}_R)) \geq 1 - \frac{2\epsilon}{3} + 1 - \frac{\epsilon}{6} + 1 - \frac{\epsilon}{6} - 2 = 1 - \epsilon$$

(2.36′) is thus proved. Similarly one proves (2.36) by applying twice the mean value theorem, by deriving with Gronwall's Lemma the estimate

$$\sup_{\substack{t \leq s \leq T \\ |\eta| \leq 2R}} \left| \frac{\partial \psi_{st}^{-1}}{\partial \eta} (\eta) \right| \leq \exp \left[\bar{\ell}(\omega) (1 + 4R^2)^2 \, \bar{T} \right]$$

with $\bar{\ell}(\omega)$ an L^P bounded r.v. as before and by using the convergence result for each $t \leq T$:

$$(\xi_{0t}^n)^{-1} (x) \xrightarrow{P} \xi_{0t}^{-1} (x)$$

uniformly in x on compact sets of $\mathbf{R}^d$ [3, p. 66]).

Proof of Proposition 2.1

Consider (2.12) and assume (d) and $u^*(t, x) \in \text{int } \mathcal{U}$ for all $(t, x) \in [0, T] \times \mathbf{R}^n$ as in (c). Then because

$$W(t, \eta) = \theta \circ \xi_T \circ \psi_t^{-1} (\eta)$$

$$\frac{\partial H (t, \eta, u, \omega)}{\partial u} \Big|_{u^*(t, \xi_t(\eta))} = 0 \text{ a.s.}$$

and because $H(t, \eta, \cdot, \omega)$ is strictly convex on the compact $\mathcal{U}$ for all (t, η) a.s. where

$$H (t, \eta, u, \omega) = \frac{\partial W}{\partial \eta} (t, \eta) \left(\frac{\partial \xi_t}{\partial \eta}\right)^{-1} (\eta) \, [f(\xi_t(\eta), u) - \tfrac{1}{2} \sum_{i=1}^{d} g_{i_x} g_i(\xi_t (\eta)) - \frac{\partial f}{\partial u} (\xi_t(\eta), u^*(t, \xi_t (\eta))) \, u]$$

we have

$$\min_{u \in \mathcal{U}} H (t, \eta, u, \omega) = H (t, \eta, u^*(t, \xi_t(\eta)), \omega) \text{ a.s.}$$

so that with $\lambda (t, \eta, \omega)$ defined by (2.10), $W(t, \eta) = \theta \circ \xi_T \circ \psi_t^{-1} (\eta)$ is a C^2 solution of the Hamilton-Jacobi-Bellman equation for (P^ω) a.s.

$$\frac{\partial W}{\partial t} + \min_{u \in \mathcal{U}} \{\frac{\partial W}{\partial \eta}(t, \eta)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\,[f(\xi_t(\eta), u) - \frac{1}{2}\sum_{i=1}^{d} g_{i_x} g_i(\xi_t(\eta))]$$

$$+ \tilde{\lambda}^T(t, \eta, \omega)\, u\} = 0$$

$$W(T, \eta) = \theta \circ \xi_T(\eta) \tag{2.40}$$

As solution of (2.40) $W(t, \eta)$ satisfies the optimality principle inequality

$$W(t_0, x) \le -\int_{t_0}^{t} \frac{\partial W}{\partial \eta}(\tau, \eta_\tau)\left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(\eta_\tau)\,\frac{\partial f}{\partial u}(\xi_\tau(\eta_\tau), u^*(\tau, \xi_\tau(\eta_\tau))\, u(\tau)\, d\tau$$

$$+ W(t, \eta_t) \text{ a.s.}$$

for any $u(t) \in \mathcal{A}$ where η_τ is the solution of (2.4) starting at t_0 from x. We also have under our assumptions (a, c) that the solution of (2.4) for $u = u^*(t, \xi_t(\eta))$ denoted η_t^*, exists, is unique, non-exploding and

$$\frac{dW(t, \eta_t^*)}{dt} = \frac{\partial W(t, \eta_t^*)}{\partial t} + \frac{\partial W}{\partial \eta}(t, \eta_t^*)\frac{d\eta_t^*}{dt}$$

$$= \frac{\partial W}{\partial \eta}(t, \eta_t^*)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t^*)\;\frac{\partial f}{\partial u}(\xi_t(\eta_t^*), u^*(t, \xi_t(\eta_t^*))) \circ u^*(t, \xi_t(\eta_t^*)) \text{ a.s.}$$

Then by the Verification theorem (see Fleming and Rishel [8, p. 87]) $u^*(t, \xi_t(\eta))$ is optimal for (P^ω) for almost all $\omega \in \Omega$ and $W(t, \eta)$ is the value function of these control problems parametrized by $\omega \in \Omega$. It remains to prove that $V(t, x) := W(t, \xi_t^{-1}(x))$ satisfies (2.13). Using now Lemmas 2.1 and 2.2 we have for each $t \in [0, T]$

$$W^n(t, (\xi_t^n)^{-1}(x)) \xrightarrow{P} W(t, \xi_t^{-1}(x))$$

$$V^n(t, x) := W^n(t, (\xi_t^n)^{-1}(x) \xrightarrow{P} \theta \circ \zeta_t^{-1}(x) = \theta \circ \zeta_T^{-1}(x)$$

uniformly in x on compacts of $\mathbf{R}^d$. As (2.13) has a unique non exploding C^2 solution (because of Kunita's existence and uniqueness theorem for SPDE's from Kunita [12, Section VI] whose conditions are met here (see (a,c)), we must have

$$V(t, x) := W(t, \xi_t^{-1}(x)) = \theta \circ \zeta_{tT}^{-1}(x)$$

because of the uniqueness of the limit in probability so that $W(t, \xi_t^{-1}(x))$ is the unique C^2 solution of (2.13) and Proposition 2.1 is proved.

We can now pass to the proof of our Lagrange multiplier theorem for anticipative control.

Proof of Theorem 2.1

As we saw in Proposition 2.1 with the Lagrange multiplier process $\tilde{\lambda}^T(t, \eta, \omega)$ defined by (2.10), the optimal control for the stochastic nonanticipative control problem (2.6), $u^*(t, \xi_t(\eta))$ is optimal for the anticipative optimal control problems (P^ω) for almost all $\omega \in \Omega$.

Due to our assumptions $\tilde{\lambda}^T(t, \xi_t^{-1}(x))$ defined according to (2.10) by

$$\tilde{\lambda}^T(t, \xi_t^{-1}(x)) := \lambda^T(t, x, \omega) = -\frac{\partial V(t, x)}{\partial x} f_u(x, u^*(t, x))$$

is $L^1(dP \times dt)$ - integrable for $x = x_t$ the solution of (0.1) corresponding to some $u_t \in \mathcal{A}$

$$E \int_0^T |\lambda^T(t, x_t, \omega)|\, dt \leq E \int_0^T |V_x(t, x_t)|\, |f_u(x_t, u^*(t, x_t)|\, dt < \infty$$

This is proved in the following way. First because f_u, θ are bounded; $V_x(t, x) = \theta_x(\zeta_t^{-1}(x))$ $\frac{\partial \zeta_t^{-1}(x)}{\partial x}$ and $x_t = \xi_t \circ \eta_t$ we see that via the estimates for flows of [15] and [3, p.39,50] :

$$\sup_{t \leq T} |\xi_t(x)| \leq k(\omega)(i + |x|^2)$$

$$\sup_{t \leq T} \left|\frac{\partial \zeta_t^{-1}}{\partial x}(x)\right| \leq l(\omega)(1 + |x|^2)$$

$\forall x \in \mathbb{R}^d$: $k(\omega).l(\omega) \in \bigcap_{p \geq 1} L^p(\Omega)$ we have that the L^1 integrability of the Lagrange multiplier comes down to showing

$$E\int_0^T k(\omega)[1 + \left(l(\omega)(1 + |\eta_t|^2)\right)^2]dt < \infty$$

and thus reduces to proving

$$\sup_{t \leq T} E|\eta_t|^{2^r} < \infty \quad \forall r \geq 1$$

This follows as usually from the existence theorem for (1.4). Consider the successive approximation sequence

$$\eta_t^0 = x_0 \, , \, \eta_t^n = x_0 + \int_0^t \left(\frac{\partial \xi_\tau}{\partial x}\right)^{-1}(\eta_\tau^{n-1})\overline{f}(\xi_\tau \circ \eta_\tau^{n-1}, u_\tau)d\tau$$

We assumed before without restricting generality deterministic initial conditions, so that

$$\sup_{t \leq T} E|\eta_t^0|^{2^r} = |x_0|^{2^r} < \infty \, .$$

By induction if

$$\sup_{t \leq T} E|\eta_t^{n-1}|^{2^r} < \infty \quad \forall r \geq 1$$

then using again the flow growth estimates in the successive approximation sequence defined above we have

$$E|\eta_t^n|^{2^r} \le A_r + B_r + B_r\int_0^t E|\eta_\tau^{n-1}|^{2^r} d\tau < \infty \tag{2.41}$$

where

$$A_r = c_r|x_0|^{2^r} \; ; \; B_r = c_r(MT)^{2^r}\left(E(l(\omega))^{2^{r+1}}\right)^{\frac{1}{2}}$$

with c_r depending on r only and M the bound for $\overline{f}$. Due to the growth estimates and the a.s. smooth flow of diffeomorphism property of $\xi_t(x)$ the coefficients of (1.4) have linear growth a.s. and are locally Lipschitz a.s. so that the existence theorem for ordinary differential equations can be applied almost surely to (1.4) : the successive approximation sequence convergences a.s. to the unique non-exploding solution of (1.4) which has thus finite moments due to (2.41) and moreover by Gronwall lemma applied to

$$E|\eta_t|^{2^r} \le A_r + B_r + B_r\int_0^t E|\eta_\tau|^{2^r} d\tau$$

we have the following estimate

$$E|\eta_t|^{2^r} \le (A_r + B_r)\exp(B_rT) \quad \forall t \in [0.T].\forall r \ge 1$$

We show next (2.11) . W (t. ξ_t^{-1}(x)) satisfies (2.14) and because of u* being an interior optimum (see (c)) for (2.6) (and thus $\frac{\partial V^*}{\partial x}(t, x)\frac{\partial f}{\partial u}(x, u^*(t, x)) = 0$) (2.7′) can be written

$$\frac{\partial V^*}{\partial t}(t, x) + \frac{\partial V^*}{\partial x}(t, x)\left[f(x, u^*(t, x)) - \frac{\partial f}{\partial u}(x, u^*(t, x))\, u^*(t, x)\right] + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 V^*}{\partial x^2} gg^T(x)\right) = 0$$
$$V^*(T, x) = \theta(x) \tag{2.42}$$

We average the integral form of (2.14) and we use Lemma 6.2.6 and Theorem 6.1.10 from [12] in our particular case to interchange expectation with differentiation and integration and to show that the stochastic integral has zero mean. We get under our asumptions

$$EV(t, x) = \theta(x) + \int_t^T \frac{\partial}{\partial x}(EV(\tau, x)) [f(x, u^*(\tau, x)) - \frac{\partial f}{\partial u}(x, u^*(\tau, x))$$

$$\times u^*(\tau, x)] d\tau + \int_t^T \frac{1}{2} \mathrm{tr} \left(\frac{\partial^2 EV}{\partial x^2} gg^T(x)\right) d\tau \qquad (2.42')$$

where we denote again $V(t, x) := W(t, \xi_t^{-1}(x))$. By the uniqueness of the solution of (2.41) (see [9,p.44])

$$V^*(t,x) = EV(t,x) = EW(t,\xi_t^{-1}(x)) \qquad (2.43)$$

and thus

$$E\lambda^T(t, x) = E\tilde{\lambda}^T(t, \xi_t^{-1}(x)) = E\left(-\frac{\partial W(t, \xi_t^{-1}(x))}{\partial x} \frac{\partial f}{\partial u}(x, u^*(t, x))\right)$$

$$= E\left(\frac{\partial V^*}{\partial x}(t, x) \frac{\partial f}{\partial u}(x, u^*(t, x))\right) = 0$$

We have proved (2.11) and the proof of Theorem 2.1 is accomplished.

Remark

Using the convexity assumption (4) the HJBSPDE (2.14) can be written

$$dV(t, x) + \min_{u \in \mathcal{U}} \{ V_x[f(x, u) - \frac{\partial f}{\partial u}(x, u^*(t, x)) u] + \frac{1}{2} \mathrm{tr}(V_{xx} gg^T(x))\} dt$$

$$+ V_x(t, x) g(x) \, dw_t = 0$$

$$V(T, x) = \theta(x)$$

where $V_x := \frac{\partial V}{\partial x}$. $V(t, x)$ is the limit of the sequence of value functions for the sequence of (deterministic) control problems indexed by $n \in \mathbf{N}$ and parametrized by $\omega \in \Omega'$

$$(P^{\omega,n}) \quad \begin{aligned} &\dot{x}_t^n = \bar{f}(x_t^n, u_t) + g(x_t^n)\, \dot{\imath}^n(t, \omega) \\ &\inf_{u \in \mathcal{M}} [\theta(x_T^n) + \int_0^T (\bar{\lambda}^n(t, x_t^n, \omega))^T u(t)dt] \end{aligned}$$

where

$$(\bar{\lambda}^n(t, x, \omega))^T = - V_x^n(t, x) \frac{\partial f}{\partial u}(x, u^*(t, x))$$

with $V^n(t, x)$ being the solution of the sequence of random HJB PDE's (2.18). Using (2.43) we have the fundamental result

$$V^*(t,x) = E\ P\text{-}\lim_{n \to \infty} V^n(t,x) = EV(t,x)$$

showing how the second order parabolic PDE of stochastic dynamic programming results by taking the limit in probability of the sequence of deterministic dynamic programming equations (2.18) (which yields the HJB SPDE (2.14)) and averaging. This is the quintessence of the relation between deterministic and stochastic optimal control generalizing to dynamic programming equations the Wong-Zakai type results [20],[18] proving how solutions of SDE's appear as limits of sequences of solutions of ordinary differential equations . Borrowing the title of Sussmann [18] ,we can say that the "gap between deterministic and stochastic" optimal control is filled by objects like HJB SPDE (2.14) . The price to pay for this result was the lengthy convergence argument in the proof of Proposition 2.1 . Moreover if $[g_i(x), g_j(x)] = 0$ for $i \le i,j \le d$ where

$$[g_i(x), g_j(x)] = \frac{\partial g_j(x)}{\partial x} g_i(x) - \frac{\partial g_i(x)}{\partial x} g_j(x)$$

(i.e. g_i commute) the value function of the pathwise optimal control problems (2.2), $V(t, x) = \theta \circ \zeta_{t,T}^{-1}(x)$ is continous with respect to the Wiener process (due to the characteristics representation this follows from the continuity w.r.t. $w(t,\omega)$ of the solution of SDE's proved in [18]) being actually the "extension by continuity" to C^0 paths (see Davis [5]) of the value function $\tilde{V}(t, x)$ of the problem

$$(\tilde{P}^{\omega}) \quad \begin{cases} \dot{x}_t = \overline{f}(x_t, u_t) + g(x_t)\, \dot{w}(t, \omega) \\ \inf\limits_{u \in \mathcal{U}} [\theta(x_T) + \int_0^T \tilde{\lambda}^T(t, x_t, \omega)\, u(t)dt] \end{cases}$$

where $\dot{w}(t, \omega)$ are C^1 noises a.s. and

$$\tilde{\lambda}^T(t, x, \omega) = - \tilde{V}_x(t, x) \frac{\partial f}{\partial u}(x, u^*(t, x))$$

Here $\tilde{V}(t, x)$ is regarded for each $\omega \in \Omega$ as a map defined on the function space $C^1([0, T], \mathbf{R}^d)$ (i.e. $\tilde{V}$ is mapping the path space).

2.3 General Lagrange multiplier formula for the problems with integral cost

If we consider optimal control problems with integral cost, Theorem 2.1 can be generalized as follows :

Theorem 2.2

Consider the following family of optimal control problems $\omega \in \Omega'$

$$\dot{\eta}_t = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta_t) \, [f(\xi_t(\eta_t), u_t) - \tfrac{1}{2} \sum_{i=1}^{d} g_{i_x} g_i(\xi_t(\eta_t))]$$

$$(\bar{P}^{\omega}) \quad \eta_0 = x_0$$

$$\inf_{u \in \mathcal{U}} [\theta \circ \xi_T(\eta_T) + \int_0^T L(t, \xi_t \circ \eta_t, u_t)\, dt + \int_0^T \tilde{\lambda}^T (t, \eta_t, \omega)\, u_t \, dt]$$

Assume (a) - (c) and

(d') $L(t, x, u)$ is continous in t, C^5 and C_b^3 in (x, u), convex in u for all t, x and

$$\{ \frac{\partial}{\partial \eta} [\theta \circ \xi_T \circ \psi_t^{-1}(\eta) + \int_t^T (L(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta), u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta))$$

$$- \frac{\partial L}{\partial u}(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta), u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta))) \, u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta))\, d\tau]\}$$

$$\times \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta) \, f_{uu}(\xi_t(\eta), u)$$

has characteristic values bounded below by $\gamma > 0$ for all $(t, \eta, u) \in [0,T] \times \mathbf{R}^d \times \mathcal{U}$ a.s.

Define

$$\tilde{\lambda}^T (t, \eta, \omega) := - \{ \frac{\partial}{\partial \eta} [\theta \circ \xi_T \circ \psi_t^{-1}(\eta) + \int_t^T (L(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta), u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta)))$$

$$- \frac{\partial L}{\partial u}(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta), u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta))) u^*(\tau, \xi_\tau \circ \psi_\tau \circ \psi_t^{-1}(\eta))\, d\tau]\}$$

$$\times\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta)\, f_u\,(\xi_t(\eta)\,,u^*(t,\xi_t(\eta))) - \frac{\partial L}{\partial u}\,(t,\xi_t(\eta),u^*(t,\,\xi_t(\eta))) \tag{2.44}$$

Then $u^*(t,\,\xi_t(\eta))$ is optimal for $(\overline{P}^\omega)$ a.s. and

$$E\,\overline{W}(t,\,\xi_t^{-1}(x)) = V^*(t,\,x)$$

$$E\,\tilde{\lambda}^T(t,\,\xi_t^{-1}(x),\,\omega) = 0$$

where $\overline{W}(t,\,\eta)$ is the value function of $(\overline{P}^\omega)$ a.s.

Proof

The proof is identical to that of Theorem 2.1 except that this time the integral cost term changes the characteristics method representation for $W(t,\,\eta)$ as follows :

$$W(t,\,\eta) = \gamma_t \circ \psi_t^{-1}(\eta) = \gamma_t \circ \psi_{tT}^{-1}\,(\eta)$$

where $\psi_{tT}^{-1}(\eta)$ is given as before by (2.39) and

$$\frac{d\gamma_t(\eta)}{dt} = -L(t,\xi_t \circ \psi_t\,(\eta),u^*(t,\xi_t \circ \psi_t(\eta))) + \frac{\partial L}{\partial u}\,(t,\,\xi_t \circ \psi_t(\eta),u^*(t,\xi_t o \psi_t(\eta)))$$

$$\times\; u^*(t,\xi_t o \psi_t(\eta))$$

$$\gamma_T(\eta) = \theta \circ \xi_T(\eta)$$

2.4 Example : nonanticipative LQG Problem solved pathwise

Consider the standard nonanticipative LQG problem

$$dx_t = (Ax_t + Bu_t)\, dt + C\, dw_t$$

(P) $$x_0 = \bar{x}_0 \tag{2.45}$$

$$\inf_{u \in \mathcal{N}} E\left[\int_0^T (x_t^T Q x_t + u_t^T R u_t)\, dt + x_T^T F x_T\right]$$

instead of solving this stochastic optimal control problem let us consider the family of deterministic problems

$$\frac{d\eta_t}{dt} = A(\eta_t + Cw_t) + Bu_t$$

$$\eta_0 = \bar{x}_0 \tag{2.46}$$

(P^ω)

$$\inf_{u \in \mathcal{M}} \left\{ \int_0^T [(\eta_t + Cw_t)^T Q(\eta_t + Cw_t) + u_t^T R u_t]\, dt + (\eta_T + Cw_T)^T F \right.$$

$$\left. \times\ (\eta_T + Cw_T) + \int_0^T \tilde{\lambda}^T(t, \eta_t, \omega)\, u_t\, dt \right\}$$

We need to determine the Lagrange multiplier $\tilde{\lambda}\ (t, \eta_t, \omega)$ such that if $u^*(t, x)$ is optimal for (2.45) then $u^*(t, \eta + Cw_t)$ is optimal for (2.46) and we have

$$V^*(t, x) = E\ W\ (t, x - Cw_t)$$

where $V^*(t, x)$ is the value function for (2.45) and $W\ (t, \eta)$ is the value function of (2.46). The decomposition of the solution x_t is here

$$x_t = \eta_t + C w_t$$

The desired value of the Lagrange multiplier is (see (2.44) and theorem 2.2) :

$$\overline{\lambda}^T(t, x, \omega) = - \frac{\partial V}{\partial x}(t, x)\, B - 2\, u^{*T}(t, x)\, R$$

where $u^*(t, x)$ is the feedback optimal control for the (nonanticipative) LQG problem

$$u^*(t, x) = - R^{-1} B^T S_t x$$

S_t being the solution of the Riccati matrix equation

$$\dot{S}_t + S_t A + A^T S_t - S_t B R^{-1} B^T S_t + Q = 0\ ,\ S_T = F$$

$V(t, x)$ is the solution of the HJBSPDE

$$dV + \min_{u \in \mathcal{U}} \{V_x(Ax + Bu) + x^T Q\, x + u^T R u + \overline{\lambda}^T(t, x, \omega)\, u + \tfrac{1}{2} \operatorname{tr}(V_{xx} C C^T)\}$$

$$\times\, dt + V_x C\, dw_t = 0 \qquad (2.17)$$

$$V(T, x) = x^T F x$$

The minimizer $\overline{u}^*(t, x, \omega)$ is

$$\overline{u}^*(t, x, \omega) = - \tfrac{1}{2} R^{-1} [B^T V_x^1 + \overline{\lambda}] \quad = - R^{-1} B^T S_t\, x = u^*(t, x)$$

We look for a solution of (2.47) of the form

$$V(t, x, \omega) = x^T S_t x + 2\beta_t^T(\omega)\, x + \gamma_t(\omega) \qquad (2.18)$$

which we plug into (2.47) and we get

$$x^T\dot{S}_t\, x\, dt + 2d\hat{\beta}_t^T x + d\gamma_t + [x^T Q x + \tfrac{1}{2}\operatorname{tr}(C^TSC) + x^T S_t Ax + 2\hat{\beta}_t^T Ax$$
$$+ x^T A^T S_t x - x^TS_tBR^{-1}B^TS^T x]\, dt + (2 x^TS_t C + 2\hat{\beta}_t^T C)\ \hat{d}w_t = 0$$

By grouping the terms

$$x^T(\dot{S}_t + S_tA + A^TS_t - S_tBR^{-1}B^TS_t + Q)\, x\, dt + x^T(2d\hat{\beta}_t + 2A^T\hat{\beta}_t\, dt$$
$$+ 2S_tC\ \hat{d}w_t) + d\hat{\gamma}_t + \tfrac{1}{2}\operatorname{tr}(C^TS_tC)\, dt + 2\hat{\beta}_t^T C\ \hat{d}w_t \equiv 0$$

we see that we must have

$$\dot{S}_t + S_tA + A^TS_t - S_tBR^{-1}B^TS_t + Q = 0$$
$$2d\hat{\beta}_t + 2\, A^T\hat{\beta}_t\, dt + 2\, S_tC\ \hat{d}w_t = 0$$
$$d\hat{\gamma}_t + \tfrac{1}{2}\operatorname{tr}(C^TS_tC)\, dt + 2\hat{\beta}_t^T C\ \hat{d}w_t = 0 \tag{2.49}$$

with terminal conditions

$$S_T = F$$
$$\hat{\beta}_T = 0$$
$$\hat{\gamma}_T = 0 \tag{2.50}$$

because

$$x^TFx \equiv x^TS_T\, x + 2\hat{\beta}_T^T(\omega)\, x + \hat{\gamma}_T(\omega)$$

in view of (2.48-2.50)

$$\overline{\lambda}^1(t, x, \omega) = -\, 2\hat{\beta}_t^T(\omega)\, B$$

independent of x due to $\dfrac{\partial f(x, u)}{\partial u} = B = \text{const}$. We recovered thus the results from Davis [5]. If we multiply the equation for $d\hat{\beta}_t$ in (2.49) by B^T to the left and we assume BB^T nonsingular we obtain a stochastic differential equation for $\overline{\lambda}^T$ (and not a stochastic partial differential equation HJB SPDE as in the general nonlinear case)

$$d\overline{\lambda}_t^T = - B^T A^T (BB^T)^{-1} B\, \overline{\lambda}_t^T\, dt - B^T S_t C\, \hat{d}w_t$$

$$\overline{\lambda}_T^T = 0$$

The robust equation of (2.47) for $W(t, \eta)$, where $W(t, x - Cw_t) = V(t, x)$, is (see (2.12))

$$\frac{\partial W}{\partial t}(t, \eta, \omega) + \min_{u \in \mathcal{U}} \{ \frac{\partial W}{\partial \eta} (A\,(\eta + Cw_t) + Bu) + \overline{\lambda}_t^T(\omega)\, u + u^T R u$$

$$+ (\eta + Cw_t)^T Q (\eta + Cw_t) \} = 0$$

$$W(T, \eta) = (\eta + Cw_T)^T F (\eta + Cw_T)$$

with solution $W(t,\eta) = (\eta + Cw_t)^T S_t (\eta + Cw_t) + 2\beta_t^T(\eta + Cw_t) + \gamma_t$

Remark

The optimal control $u^*(t,x)$ is not bounded but many of the assumptions of our results can be weakened: $\mathcal{U}$ need not be compact as we use here the fact that the expression to be minimized in HJB PDE is quadratic in u; f can be linear in x as $\xi_t(x) = x + Cw_t$ so that $\frac{\partial \xi_t}{\partial x}(x) = I_d$ and the linear growth in x of $a(t,x,\omega)$ holds a.s. ensuring the nonexplosion of $\psi_t(\eta)$ in (2.7); the characteristics method works with θ having Lipschitz continous derivatives so in particular with θ linear etc.

3 : Stochastic anticipative optimal control and almost sure (pathwise) optimal control

We consider now the anticipative stochastic control problem

$$
(P^{o}) \quad
\begin{aligned}
& dx_t = \overline{f}\,(x_t, u_t)dt + g(x_t) \circ dw_t \\
& x_0 = \overline{x}_0 \\
& \inf_{u \in \mathcal{A}} E[\theta(x_T)] \qquad (3.1) \\
& x_t \in \mathbb{R}^d,\ g(x) = [g_1(x) \vdots \cdots \vdots g_p(x)] \\
& f(x) = f(x) - \tfrac{1}{2} \sum_{i=1}^{p} g_{i_x}(x)\, g_i(x)
\end{aligned}
$$

to which we associate a family of deterministic optimal control problems $(P^{0,\omega})_{\omega \in \Omega'}$ via the SDE solution decomposition formula (1.2) $x_t(x_0) = \xi_t \circ \eta_t(\bar{x}_0)$ which is our definition of solution of (1.1) for $u_t \in \mathcal{A}$:

$$
(P^{o,\omega})_{\omega \in \Omega'} \quad
\begin{aligned}
& \frac{d\eta_t(x_o)}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta_t(x_0))\overline{f}\,(\xi_t \circ \eta_t(x_0)\ ,u_t) \\
& \eta_t(\bar{x}_0) = \overline{x}_0 \qquad (3.2) \\
& \inf_{u \in \mathcal{M}} [\theta \circ \xi_T(\eta_T(x_0))]
\end{aligned}
$$

where $\mathcal{M}$ is the class of measurable functions $u:[0,T] \to \mathcal{U}, \mathcal{U}$ a compact in $\mathbf{R}^m$. The relation between (P^0) and $(P^{0,\omega})_{\omega \in \Omega'}$ is

$$\inf_{u \in \mathcal{A}} E[\theta(x_T)] = E\left(\inf_{u \in \mathcal{M}} (\theta \circ \xi_T(\eta_T(\bar{x}_0)))\right)$$

provided again that the infimum on the right is attained for each ω and the function assigning to each ω the minimizer of $P^{o,\omega}$ is measurable, which will be the case under our assumptions as we use a measurable selection . This shows that we can solve (P^o) by solving $(P^{o,\omega})$ for $\omega \in \Omega$ and averaging the cost over the sample space, Ω.

We make the following assumptions

(a′) f is C_b^4 in (x,u) (i.e. with bounded mixed derivatives up to order 4) and bounded, g_i are C_b^5 and bounded for i=1,...,p, θ is C_b^5

(c′) the nonanticipative stochastic optimal control problem

$$dx_t = \bar{f}(x_t,u_t)dt + g(x_t) \circ dw_t$$

$$x_0 = x_0$$

$$\inf_{u \in \mathcal{N}} E[\theta(x(T))]$$

has a feedback solution $u^*(t,x)$ continuous in (t, x), C_b^2 in x so that, under (a) its value function is the $C^{1,2}$ unique solution of

$$\frac{\partial V^*}{\partial t} + \min_{u \in \mathcal{U}} \left\{ \frac{\partial V^*}{\partial x} \; f(x,u) \right\} + \frac{1}{2} \operatorname{tr}\left(\frac{\partial^2 V^*}{\partial x^2} \, gg^T \right) = 0,$$

$$V^*(T,x) = \theta(x) .$$

(The smoothness assumptions on $u^*(t,x)$ can be relaxed if we impose in addition the uniform parabolicity of gg^T [8,p.129])

(e) Consider $Y(z,x) = \min_{u \in \mathcal{U}} \{ z^T \bar{f}(x,u) \}$, $z,x \in \mathbf{R}^d$. Selection lemmas exist (see Beneš [2, Lemma 5]) which give a measurable or even a C^1 minimizer $\phi(z,x)=u^0(z,x)$ such that $Y(z,x) = z^T \bar{f}(x,\phi(z,x))$. The implicit function theorem can be used to find conditions for smoother minimizers. We assume here that ϕ is C_b^4 in (x,z).

Corresponding to each member of the family of problems (3.2) we have a random HJB PDE (i.e. PDE with random coefficients) for the value function $W(t,\eta) := \inf_{u_\omega \in \mathcal{M}} [\theta \circ \xi(T, t, \eta))]$:

$$\frac{\partial W}{\partial t} + \min_{u \in \mathcal{U}} \left\{ \frac{\partial W}{\partial \eta} \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta) \bar{f} (\xi_t(\eta), u) \right\} = 0 \tag{3.3}$$

$$W(T,\eta) = \theta \circ \xi_T(\eta).$$

Using (e) we get under our assumptions a minimizer in the form

$$u^0(t,\eta,\omega) = \phi\left(\frac{\partial W}{\partial \eta} (t,\eta) \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta), \xi_t(\eta)\right).$$

which substituted in (3.3) yields

$$\frac{\partial W}{\partial t} (t,\eta) + \frac{\partial W}{\partial \eta} (t,\eta) \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta) \bar{f} \left(\xi_t(\eta), \phi\left(\frac{\partial W}{\partial \eta} (t,\eta) \left(\frac{\partial \xi_t}{\partial x}\right)^{-1} (\eta), \xi_t(\eta)\right)\right) = 0 \tag{3.4}$$
$$W(T,\eta) = \theta \circ \xi_T(\eta)$$

Using from (1.2) $\eta = \xi_t^{-1}(x)$ we get the expression of the value function in terms of x, the initial value at time t for (1.1)

$$V^0(t,x) := W(t, \xi_t^{-1}(x))$$
$$\hat{u}^0(t,x,\omega) := u^0(t,\xi_t^{-1}(x),\omega)$$

and it will be shown that $V^0(t,x)$ is the unique global C^2 solution of the (backward) stochastic PDE (Hamilton-Jacobi-Bellman stochastic PDE)

$$\begin{cases} dV^0(t,x) + \frac{\partial V^0}{\partial x}(t,x)\bar{f}\left(x,\phi\left(\frac{\partial V^0}{\partial x}(t,x),x\right)\right)dt + \frac{\partial V^0}{\partial x}(t,x)g(x)\hat{o}\, dw_t = 0 \\ V^0(T,x) = \theta(x) \end{cases} \tag{3.5}$$

This solution is $V^0(t,x) = \inf_{u_\omega \in \mathcal{M}} \mathcal{J}[x(T:t,x)]$ and is expressed by the stochastic characteristics method as $V^0(t,x) = \nu_t \circ \varphi_t^{-1}(x)$ where

$$d\varphi_t(x) = [f(\varphi_t(x), \phi(\chi_t(x), \varphi_t(x))) + \chi_t^T(x) f_u(\varphi_t(x), \phi(\chi_t(x), \varphi_t(x)))\, \phi_\chi(\chi_t(x), \varphi_t(x))]dt$$
$$+ g(\varphi_t(x)) \hat{o}\, dw_t; \quad \varphi_T(x) = x \tag{3.6}$$

$$d\chi_t(x) = -[f_x(\varphi_t(x), \phi(\chi_t(x), \varphi_t(x))) + f_u(\varphi_t(x), \phi(\chi_t(x), \varphi_t(x)))\phi_\varphi(\chi_t(x), \varphi_t(x))]^T \chi_t(x)dt$$
$$- \sum_{i=1}^{d} g_{i,x}^T(\varphi_t(x))\, \chi_t(x) \hat{o}\, dw_t^i; \quad \chi_T(x) = \theta_x(x) \tag{3.7}$$

$$\frac{d\nu_t}{dt}(x) = \chi_t^T(x)\, f_u(\varphi_t(x), \phi(\chi_t(x), \varphi_t(x)))\phi_\chi(\chi_t(x), \varphi_t(x))\chi_t(x)$$
$$\nu_T(x) = \theta(x) \tag{3.8}$$

where $f_u = \frac{\partial f}{\partial u}$ is the $n \times m$ matrix of components $f_u^{ij} = \frac{\partial f_i}{\partial u_j}$, $\phi = \phi(\chi, \varphi)$ $\phi_\varphi = \frac{\partial \phi}{\partial \varphi}$ etc. "$\hat{o}$" is the Stratonovich backward differential notation and $(\varphi_t(x), \chi_t(x))$ are defined as in (1.8),(1.9) using the flows of (3.6),(3.7) with general terminal condition. Making the notations

$$F(\varphi, \chi) = \chi^T f(\varphi, \phi(\chi, \varphi))$$
$$G(\varphi, \chi) = \chi^T g(\varphi)$$

we can rewrite (3.5) as

$$\begin{cases} dV^0(t,x) + F(x, \frac{\partial V^0}{\partial x}(t,x))\, dt + G\left(x, \frac{\partial V^0}{\partial x}(t,x)\right) \hat{o}\, dw_t = 0 \\ V^0(T,x) = \theta(x) \end{cases} \tag{3.9}$$

and (3.6)-(3.8) as

$$d\varphi_t(x) = F_\chi(\varphi_t(x), \chi_t(x))dt + G_\chi(\varphi_t(x), \chi_t(x)) \hat{o}\, dw_t \tag{3.10}$$

$$d\chi_t(x) = -F_\varphi(\varphi_t(x), \chi_t(x))dt - G_\varphi(\varphi_t(x), \chi_t(x)) \hat{o}\, dw_t \tag{3.11}$$

$$\frac{d\nu_t}{dt}(x) = -(F(\varphi_t(x), \chi_t(x)) - F_\chi(\varphi_t(x), \chi_t(x))\chi_t(x)) \tag{3.12}$$

$$\varphi_T(x) = x, \; \eta_T(x) = \theta(x), \;\; \chi_T(x) = \theta_x(x)$$

(3.9) is a nonlinear SPDE which has only a local solution for $t \in]\tau(\omega),T]$,(i.e. down to a stopping time $\tau(\omega)$; [10, Chapter 6]) because $\varphi_t(x)$ is only a local flow of diffeomorphisms a.s. up to a stopping time due to the coupling with $\chi_t(x)$. We will impose conditions ensuring that $\varphi_t(x)$ is a global flow of diffeomorphisms a.s. by making the stochastic Hamiltonian system (3.10) - (3.11) admit a certain invariant Lagrangian submanifold or random conservation law $\chi_t(x)=\vartheta(t,\varphi_t(x),\omega)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ a.s.. which will decouple (3.10) from (3.11).

Equation (3.9) has (3.4) as a robust random PDE (PDE family parametrized by $\omega \in \Omega'$) with characteristics solution $W(t,\eta) = \gamma_t \circ \psi_t^{-1}(\eta)$ where

$$\frac{d\psi_t}{dt} = \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t)\bar{f}\left(\xi_t(\psi_t), \sigma(\delta_t^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t), \xi_t(\psi_t))\right)$$

$$+ \left(\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t)\right)^T \sigma_\chi^T \cdot \bar{f}_u^T\left(\xi_t(\psi_t), \sigma(\delta_t^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t), \xi_t(\psi_t))\right)$$

$$\times\left(\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t)\right)^T \delta_t =: \bar{F}_\delta^\omega(t,\psi_t,\delta_t)$$

$$\psi_T = \eta \tag{3.13}$$

$$\frac{d\delta_t}{dt} = -\left(\bar{f}_x \frac{\partial \xi_t}{\partial x}(\psi_t) + \bar{f}_u\left(\sigma_\varphi \frac{\partial \xi_t}{\partial x}(\psi_t) + \sigma_\chi \frac{\partial}{\partial x}\left(\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}\right)^T(\psi_t)\delta_t\right)\right)^T$$

$$\times \left(\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t)\right)^T \delta_t - \bar{f}^{\,T} \frac{\partial}{\partial x}\left(\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}\right)^T (\psi_t)\delta_t = -\bar{F}^{\omega}_{\psi}(t,\psi_t,\delta_t)$$

$$\delta_T = \theta_x(\xi_T(\eta)) \frac{\partial \xi_T}{\partial x}(\eta) \tag{3.13'}$$

$$\frac{d\gamma_t}{dt} = \delta_t^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi_t) f_u\, \phi_\chi \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}\delta_t =: -\left(\bar{F}^{\omega}(t,\psi_t,\delta_t) - \bar{F}^{\omega}_{\delta}(t,\psi_t,\delta_t)\delta_t\right) \tag{3.13''}$$

$$\gamma_T = \theta \circ \xi_T(\eta)$$

with

$$\bar{F}^{\omega}(t,\psi_t,\delta_t) := \delta^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi)\bar{f}\left(\xi_t(\psi),\phi\left(\delta^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi),\xi_t(\psi)\right)\right) = F(\varphi,\chi)\Big|_{\substack{\varphi=\xi_t(\psi)\\ \chi=\delta^T(\partial\xi_t/\partial x)^{-1}(\psi)}}$$

We omit η from the notation here, writing $(\psi_t, \delta_t, \gamma_t)$ instead of $(\psi_t(\eta), \delta_t(\eta), \gamma_t(\eta))$ defined as in (1.8)-(1.10) using the flow of (3.13),(3.13') with general terminal condition. We also omitted for simplicity the variables of functions in the second and third equation. Considering

$$\frac{d\psi^{-1}_{ts}(\eta)}{ds} = \left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(\psi^{-1}_{ts})\bar{f}\left(\xi_s(\psi^{-1}_{ts}),\phi\left(\delta_s^T\left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(\psi^{-1}_{ts}),\ \xi_s(\psi^{-1}_{ts})\right)\right) + \left(\left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(\psi^{-1}_{ts})\right)^T$$

$$\times \phi_\chi^T f_u^T\left(\left(\frac{\partial \xi_s}{\partial x}\right)^{-1}(\psi^{-1}_{ts})\right)^T \delta_s := \bar{F}^{\omega}_{\delta}\left(s,\psi^{-1}_{ts}(\eta),\delta_s(\eta)\right) \tag{3.14}$$

$$\psi^{-1}_{tt}(\eta) = \eta \qquad\qquad t \le s \le T$$

we have

$$\psi_t^{-1}(\eta) = \psi^{-1}_{tT}(\eta) \ .$$

3.1 Main results on anticipative control and on the cost of perfect information

The first results concern the optimal control and optimal cost function for the anticipative stochastic control problem (P^{o}) solved by means of $(P^{o,\omega})_{\omega \in \Omega'}$ with Ω' as in Definition 1.

Theorem 3.1 Assume (a′), (e) and

(f) $\{F(\varphi,\chi),\ \chi_i - \theta_{x_i}(\varphi)\} = 0$

$\{G^{\ell}(\varphi,\chi),\ \chi_i - \theta_{x_i}(\varphi)\} = 0$

for $(\varphi,\chi) \in L=\{(\varphi,\chi) \in \mathbb{R}^{2d} | \chi=\theta_x(\varphi)\}$, $\ell=1,\dots,p$, $i=1,\dots,d$ where $\{\cdot,\cdot\}$ is the Poisson bracket

defined by $\{h(\varphi,\chi),k(\varphi,\chi)\} = \sum_{i=1}^{d} \left(\frac{\partial k}{\partial \chi_i}\frac{\partial h}{\partial \varphi_i} - \frac{\partial k}{\partial \varphi_i}\frac{\partial h}{\partial \chi_i}\right)$.

Then $\tilde{u}^0(t,x,\omega) := u^0(t,\xi_t^{-1}(x),\omega) = \arg \inf_{u \in \mathcal{A}} E[\theta(x(T;t,x))] = \phi(\theta_x(x),x))$ (i.e. the optimal control is a feedback control) and

$$V^o(t,x,\omega) = \inf_{u_\omega \in \mathcal{M}} \theta(x(T;t,x)) = \theta(x)+F(0,\theta_x(0))(T-t) + \sum_{\ell=1}^{p} G^{\ell}(0,\theta_x(0))$$

$$\times [w_i(T,\omega) - w_i(t,\omega)].V(t,x) = \inf_{u \in \mathcal{A}} E[\theta(x(T;t,x))] = \theta(x)+F(0,\theta_x(0))(T-t) = EV^0(t,x,\omega)$$

where $V^0(t,x,\omega)$ is the unique, global C^2 solution of HJBSPDE

$$dV^0(t,x) + \min_{u \in \mathcal{U}}\left\{\frac{\partial V^0}{\partial x}(t,x)\,\bar{f}(x,u)\right\}dt + \frac{\partial V^0}{\partial x}(t,x)g(x)\circ dw_t = 0$$

$$V^0(T,x) = \theta(x) \tag{3.15}$$

Remark L is a Lagrangian submanifold as the symplectic form $\sum_{i=1}^{d} d\chi_i \wedge d\varphi_i$ is null on L(Arnold [1]).

The next theorem characterizes the optimal control and optimal cost function of (P^0) in another case in which these can be globally defined (i.e., for $t \in [0,T]$, $x \in \mathbf{R}^d$). In this case the optimal control may be anticipative and $V^0(t,x,\omega)$ is not separable in t and x as it is in Theorem 3.1.

Theorem 3.2 Assume (a'), (c) and

(f') $\{F_{\chi_i}(\varphi,\chi), F(\varphi,\chi)\} = 0 \quad \{F_{\chi_i}(\varphi,\chi), G^{\ell}(\varphi,\chi)\} = 0$

for $(\varphi,\chi) \in \mathbf{R}^{2d}$, $\ell=1,\dots,p$; $i=1,\dots,d$

(f'') $\{F_{\chi_i}(\varphi,\chi), \chi_j - \theta_{x_j}(\varphi)\} = 0$

for $(\varphi,\chi) \in L$

Then $\tilde{u}_0(t,x,\omega) = \phi(\chi_t \circ \varphi_t^{-1}(x),x)$ and $V^0(t,x,\omega) = \nu_t \circ \varphi_t^{-1}(x)$ is the unique, global, C^2 solution of (3.15) where

$$d\varphi_t(x) = F_\chi(0,\theta_x(0))dt + g(\varphi_t(x))\hat{\circ}\, dw_t, \; \varphi_T(x) = x \qquad (3.16)$$

and $\nu_t(x)$, $\chi_t(x)$ are given by (3.7), (3.8) in which $\varphi_t(x)$ is the flow of (3.16) above.

Example

It is interesting to see how these conditions and formulas look in the particular case of the deterministic LQG problem

$$\dot{x}_t = Ax_t + Bu_t, \; x_0 = \bar{x}_0$$

$$\inf_{u \in \mathcal{U}} \left[\int_0^T (x_t^T Q x_t + u_t^T R u_t)dt + x_T^T F x_T \right]$$

We have the HJB PDE (dynamic programming equation after minimizing)

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}Ax - \frac{1}{4}\frac{\partial V}{\partial x} BR^{-1}B^T\left(\frac{\partial V}{\partial x}\right)^T + x^TQx = 0$$

$$V(T,x) = x^TMx.$$

Condition (f) amounts to $\{F(\varphi,\chi), \chi_i-2(M\varphi)_i\} = 0$ on the Lagrangian submanifold $L = \{\varphi,\chi) \in \mathbf{R}^{2d} | \chi-2M\varphi=0\}$, where $F(\varphi,\chi) = \chi^TA\varphi - \frac{1}{4}\chi^T BR^{-1}B^T\chi+\varphi^TQ\varphi$ is the Hamiltonian, $i=1,\dots,d$ and $(M\varphi)_i$ denotes the i-th component. After calculations we get that (f) is in fact

$$Q + MA + A^TM - MBR^{-1}B^TM = 0$$

i.e. M must satisfy the algebraic Riccati matrix equation and thus is a stationary solution of the Riccati matrix differential equation. We get $V(t,x) = x^TMx$ (stationary cost function). If M_t solves instead the differential Riccati matrix equation (DRE)

$$-\dot{M}_t = Q + M_tA + A^TM_t - M_tBR^{-1}B^TM_t, \; M_T = M$$

then

$$V(t,x) = x^TM_tx$$

because DRE for M_t amounts to the condition for time dependent integrals of motion for stochastic Hamiltonian systems [3, p. 231]

$$\frac{d}{dt}(\chi_i - 2(M_t\varphi)_i) + \{F, \chi_i - 2(M_t\varphi)_i\} = 0 \text{ on } L_t = (\varphi,\chi) \in \mathbf{R}^{2d} | \chi - 2M_t\varphi=0\}$$

which is the generalization of (f)($\{F(\varphi,\chi),\chi - 2M\varphi\}$ is the vector with the ith component being $\{F(\varphi,\chi),\chi_i - 2(M\varphi)_i\}$).Indeed

$$\frac{d}{dt}(\chi - 2M_t\varphi) + \{\chi^T A\varphi - \tfrac{1}{4}\chi^T BR^{-1}B^T\chi + Q\varphi,\chi - 2M_t\varphi\} = 0$$

on L_t .This can be seen by first calculating the Poisson bracket and differentiating with respect to time to get

$$-2\dot{M}_t\varphi + (-2M_t)(A\varphi - \tfrac{1}{2}BR^{-1}B^T\chi) - A^T\chi - 2Q\varphi = 0.$$

We then substitute $\chi=2M_t\varphi$ and factor out 2φ to get

$$-\dot{M}_t = M_tA + A^TM_t - M_tBR^{-1}B^TM_t + Q,\ M_T = M.$$

We state now the generalizations of Theorem 3.1 obtained by imposing conditions for the existence of time varying deterministic and random conservation laws .

<u>Theorem 3.3</u>

Assume (a′), (e) and assume there exist $\beta_1(t,\varphi),\ldots,\beta_d(t,\varphi)$ which are C_b^3 in φ and C^1 in t satisfying $\forall(\varphi,\chi) \in L_t = \{(\varphi,\chi) \in \mathbf{R}^{2d} | \chi - \beta(t,\varphi) = 0\}\ \forall t \in [0,T]$

$$\frac{\partial\beta_i}{\partial t}(t,\varphi) + \{F(\varphi,\chi),\ \chi_i - \beta_i(t,\varphi)\} = 0 \qquad i=1,\ldots,d.$$

$$\text{(g)} \qquad \{G^\ell(\varphi,\chi),\ \chi_i - \beta_i(t,\varphi)\} = 0 \qquad i=1,\ldots,d;\ \ell=1,\ldots,p$$

$$\beta_i(T,\varphi) = \theta_x(\varphi);$$

Then $u^0(t,x) = \phi(\beta(t,x),x)$ and $V^0(t,x,\omega) = \nu_t \circ \varphi_t^{-1}(x)$ is the unique, C^2 global solution of (3.5) where

$$d\varphi_t(x) = F_\chi(\varphi_t(x),\ \beta(t,\varphi_t(x))dt + G_\chi(\varphi_t(x),\beta(t,\varphi_t(x))) \circ dw_t \qquad (3.17)$$

$$\varphi_T(x) = x$$

$$\frac{d\nu_t}{dt}(x) = -\left(F(\varphi_t(x), \beta(t,\varphi_t(x))) - F_\chi(\varphi_t(x), \beta(t,\varphi_t(x)))\right)\beta^T(t,\varphi_t(x))) \tag{3.17'}$$

$$\eta_T(x) = \theta(x)$$

and $\{\ell(\varphi,\chi,t), k(\varphi,\chi,t)\} = \sum_{i=1}^{d}\left(\frac{\partial k}{\partial \chi_i}\frac{\partial \ell}{\partial \varphi_i} - \frac{\partial k}{\partial \varphi_i}\frac{\partial \ell}{\partial \chi_i}\right)$ for each $t \in [0,T]$.

Remarks

1) Assume $\beta_i(t,\varphi) = \dfrac{\partial\gamma_i(t,\varphi)}{\partial\varphi}$ (i.e. β_i are gradients of some γ_i, for t fixed) exist satisfying (g). Then $V^o(t,x,\omega)$ can be expressed as

$$V^0(t,x,\omega) = \gamma(t,x) + \xi(t,\omega)$$

$$d\xi(t,\omega) = \xi^1(t)dt + \sum_{\ell=1}^{p}\xi_\ell^2(t)\hat{d}w_t^\ell\ ,\ \xi(T,\omega) = 0 \tag{3.18}$$

where $\hat{d}$ is the backward Ito differential notation and with ξ^1, ξ_ℓ^2 given by

$$\xi^1(t) = -\frac{\partial\gamma}{\partial t}(t,\varphi) - F\left(\varphi, \frac{\partial\gamma(t,\varphi)}{\partial\varphi}\right)$$

$$\xi_\ell^2(t) = -\ G^\ell\left(\varphi, \frac{\partial\gamma(t,\varphi)}{\partial\varphi}\right) \tag{3.19}$$

If we consider stochastic time-varying prime integrals of the stochastic characteristic system of (3.5) of the form $\chi_t(x) = \beta(t,\varphi_t(x),\omega)$ we get an even more general form of Theorem 3.2. We denote $D_x^k := \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$, $|k| = k_1 + \cdots + k_n$

Theorem 3.4

Assume (a'), (e) and assume there exist d gradient backward random fields $\beta_1(t,\varphi,\omega), \ldots, \beta_d(t,\varphi,\omega)$ (i.e. $\exists, \gamma_i(t,\varphi,\omega)$ s.t. $\beta_i(t,\varphi,\omega) = \partial\gamma_i(t,\varphi,\omega)/\partial\varphi$) which are C^3 in φ, $\forall$ $t \in [0,T]$ a.s. ,

$|D_x^k \beta_j(t,x,\omega)| \le r(t,\omega) \in \mathbf{L}^1_{loc}([0,T])\ \forall\, x \in \mathbf{R}^d$ a.s. for $|k| \le 3$ (i.e. for some processes $r(t,\omega)$ almost sure locally integrable on $[0,T]$) and have differentials given by

$$d\beta_j(t,\varphi,\omega) = \beta_j^1(t,\varphi,\omega)dt + \sum_{\ell=1}^{p} \beta^2_{j,\ell}(t,\varphi,\omega) \circ dw_t^\ell \quad ; \quad j=1,\dots,d \tag{3.20}$$

satisfying for all $(\varphi,\chi) \in L_t(\omega)=\{(\varphi,\chi) \in \mathbf{R}^{2d} \,|\, \chi - \beta(t,\varphi,\omega)=0\}$ and for all ω in some Ω'' with $P(\Omega'')=1$:

$$\beta_i^1(t,\varphi,\omega) + \{F(\varphi,\chi),\ \chi_i - \beta_i(t,\varphi,\omega)\} = 0$$

(g')

$$\beta^2_{i,\ell}(t,\varphi,\omega) + \{G^\ell(\varphi,\chi),\ \chi_i - \beta_i(t,\varphi,\omega)\} = 0 \qquad i=1,\dots,d;\ \ell=1,\dots,p.$$

$$\beta_j(T,\varphi,\omega) = \theta_x(x)$$

Then $u^0(t,x,\omega)=\phi(\beta(t,x,\omega),x))$ is F^t_T -adapted and $V^0(t,x,\omega) = \gamma(t,x,\omega) + \bar{\xi}(t,\omega)$, $\bar{\xi}(T,\omega) = 0$ with $\bar{\xi}(t,\omega)$ given by

$$d\bar{\xi}(t,\omega) = \bar{\xi}^{\,1}(t,\omega)dt + \sum_{\ell=1}^{p} \bar{\xi}^{\,2}_\ell(t,\omega) \circ dw_t^\ell$$

$$\gamma^1(t,\varphi,\omega) + \bar{\xi}^{\,1}(t,\omega) + F\left(\varphi, \frac{\partial\gamma(t,\varphi,\omega)}{\partial\varphi}\right) = 0 \quad \text{a.s.} \tag{3.21}$$

$$\gamma^2_\ell(t,\varphi,\omega) + \bar{\xi}^{\,2}_\ell(t,\omega) + G^\ell\left(\varphi, \frac{\partial\gamma(t,\varphi,\omega)}{\partial\varphi}\right) = 0 \quad \text{a.s.} \quad l=1,\dots,p$$

$$d\gamma(t,\varphi,\omega) = \gamma^1(t,\varphi,\omega)dt + \sum_{\ell=1}^{p} \gamma^2_\ell(t,\varphi,\omega) \circ dw_t^\ell$$

where $\dfrac{\partial\gamma^1(t,\varphi,\omega)}{\partial\varphi} = \beta^1(t,\varphi,\omega)$, $\dfrac{\partial\gamma^2_\ell}{\partial\varphi}(t,\varphi,\omega) = \beta^2_\ell(t,\varphi,\omega)$ a.s.

$V^0(t,x,\omega)$ is the unique C^2 global solution of (3.5) and it can be represented by $V^0(t,x,\omega) = \nu_t \circ \varphi_t^{-1}(x)$

with $\nu_t(x),\varphi_t(x)$ given by (3.17'),(3.17'') where instead of $\beta(t,\varphi_t(x))$ we substitute $\beta(t,\varphi(x),\omega)$ defined above.

In his pioneering work on stochastic Hamiltonian mechanics [3], Bismut was the first to consider deterministic conservation laws for stochastic Hamiltonian systems . We defined here a generalization of these invariants which in the case of stochastic systems must naturally be random conservation laws.

We can now state the results concerning the "cost of perfect information": the difference between the nonanticipative optimal cost and the optimal cost when anticipative controls are allowed. In general we have

Theorem 3.5

Assume (a'),(c'),(e) and assume that for all $x \in \mathbf{R}^d$ $\varphi_t(x)$, the stochastic flow of (3.6), is a global flow of diffeomorphisms for $t \in [0,T]$ almost surely (which is true under the assumptions of any of Theorems 3.1 - 3.4). Then

$$\Delta(t,x) = V^*(t,x) - EV^0(t,x) = \int_t^T E[\lambda_s \circ \varphi_s^{-1}(x^*(s;t,x))(f(x^*(s;t,x),\ u^*(s,x^*(s;t,x))) - f(x^*(s;t,x),\phi(\lambda_s \circ \varphi_s^{-1}(x^*(s;t,x)),x^*(s;t,x))))]ds \qquad \text{for any } t \in [0,T],\ x \in \mathbf{R}^d \tag{3.22}$$

where $x^*(s;t,x)$ is the solution of

$$dx_s = f(x_s,u^*(s,x_s))ds + g(x_s)dw_s\ ,\ x_t = x \qquad s \in [t,T]$$

In the case when the system of stochastic characteristics (3.6) - (3.8) admits a random conservation law $\beta(t,x,\omega)$, which is the case of Theorem 3.4, we have in particular the following formula for the cost of information on the future

Corollary 3.1 Under the assumptions of Theorem 3.4 and under (c') we have

$$\Delta(t,x)=V^*(t,x)-EV^0(t,x)=\int_t^T E[\beta(s,x^*(s;t,x),\omega)\Big(f(x^*(s;t,x),u^*(s,x^*(s;t,x)))$$

$$-f(x^*(s;t,x),\phi(\beta(s,x^*(s;t,x),\omega),x^*(s;t,x)))\Big)]ds \qquad (3.22')$$

Similar corollaries hold for each of the Theorems 3.1, 3.2, 3.3 by replacing in (3.22′) $\beta(s,\cdot,\omega)$ by the corresponding formula for $\frac{\partial V^0}{\partial x}(s,\cdot,\omega)$. In the case when (f) or (g) hold we can show that the cost of perfect information is zero and the optimal anticipative control is nonanticipative (feedback).

<u>**Corrolary 3.2**</u> Assume that either (f) or (g) hold . Then $\Delta(t,x)=0$

3.2 Proof of the main results

Proof of Theorems 3.1 - 3.4

In order to prove Theorems 3.1 - 3.4 we will follow the same method as in §2.2. We consider the pathwise problems $(P^{0,\omega})_{\omega \in \Omega}$ and random PDE's (3.3) which after applying the selection lemma (see (e)) become (3.4). Using the Verification Theorem [8, p. 87] and the characteristics method (3.13) for $W(t,\eta)$, the minimizer $u^0(t,\eta,\omega)$ is optimal for $(P^{o,\omega})$ and $W(t,\eta)$ is the value function. $W(t,\eta) = \gamma_t \circ \xi_t^{-1}(\eta)$ (see (3.13), (3.14)) and

$$\frac{dW(t,\eta_t^0)}{dt} = \frac{\partial W}{\partial t}(t,\eta_t^0) + \frac{\partial W}{\partial \eta}(t,\eta_t^0)\frac{d\eta_t^0}{dt} = \frac{\partial W}{\partial t}(t,\eta_t^0) + \frac{\partial W}{\partial \eta}(t,\eta_t^0)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t^0)\bar{f}\Big(\xi_t(\eta_t^0)$$

$$\phi\Big(\frac{\partial W}{\partial \eta}(t,\eta_t^0)\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\eta_t^0),\xi_t(\eta_t^0)\Big)\Big) = 0 \qquad \text{a.s.}$$

where η_t^0 is η_t as given by (1.4) for $u_t = u^0(t,\eta,\omega)$.

To show that $W(t,\xi_t^{-1}(x)) := V^0(t,x) = \inf_{u_\omega \in \mathcal{M}_b} \theta(x(T,t,x))$ is the unique global solution of (3.5) we consider linear interpolation approximations $\{\dot{w}^n(t,\omega)\}_{n \in \mathbf{N}}$ (2.17) of $w(t,\omega)$ in

$$\frac{d\xi_t^n}{dt} = g(\xi_t^n)\dot{w}^n(t,\omega)$$

Corresponding to this we have a sequence of random PDE's

$$\frac{\partial W^n}{\partial t}(t,\eta) + \frac{\partial W^n}{\partial \eta}(t,\eta)\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}(\eta)\,\bar{f}\left(\xi_t^n(\eta),\phi\Big(\frac{\partial W^n}{\partial \eta}(t,\eta)\left(\frac{\partial \xi_t^n}{\partial x}\right)^{-1}(\eta),\xi_t^n(\eta)\Big)\right) = 0$$

$$W^n(T,\eta) = \theta \circ \xi_T^n(\eta) \tag{3.23}$$

Then by ordinary differential calculus $W^n(t,(\xi_t^n)^{-1}(x)) := V^{0,n}(t,x)$ satisfies

$$\frac{\partial V^{o,n}(t,x)}{\partial t} + \frac{\partial V^{o,n}}{\partial x}(t,x)\bar{f}\left(x,\phi\Big(\frac{\partial V^{o,n}}{\partial x}(t,x),x\Big)\right) + \frac{\partial V^{o,n}}{\partial x}(t,x)\,g(x)\,\dot{w}_t^n = 0$$

$$V^{o,n}(T,x) = \theta(x) \tag{3.24}$$

As in §2.2 we can prove the following convergence results in probability for each t uniformly in x and η

$$V^{o,n}(t,x) \xrightarrow{P} V^{o}(t,x) \tag{3.25}$$

$$W^{n}(t,\eta) \xrightarrow{P} W(t,\eta) \tag{3.26}$$

$$W^{n}(t,(\xi_t^n(x))^{-1}) \xrightarrow{P} W(t,\xi_t^{-1}(x)). \tag{3.27}$$

where V^o and W are the solutions of (3.5) and (3.4). (3.25), for example, can be shown using the stochastic and ordinary characteristics SDE representations to reduce (21) to show the convergence of the solutions of characteristics SDE's. In our notation

$$V^{o,n}(t,x) = \nu_t^n \circ (\varphi_t^n)^{-1}(x) \xrightarrow{P} V^{o}(t,x) = \nu_t \circ \varphi_t^{-1}(x). \tag{3.28}$$

where for example under the assumptions of Theorem 3.1, as we will see, we have

$$\frac{d\varphi_t^n}{dt}(x) = F_\lambda(\varphi_t^n(x), \theta_x(\varphi_t^n(x))) + g(\varphi_t^n(x))\dot{v}_t^n,\ \varphi_T^n(x) = x \tag{3.29}$$

$$\frac{d\nu_t^n}{dt}(x) = \theta_x^T(\varphi_t^n(x))\frac{\partial f}{\partial u}(\varphi_t^n(x),\phi(\theta_x(\varphi_t^n(x)),\varphi_t^n(x)))$$

$$\times \frac{\partial\phi}{\partial\lambda}(\theta_x(\varphi_t^n(x)),\varphi_t^n(x))\theta_x(\varphi_t^n(x)),\ \nu_T^n(x) = \theta(x) \tag{3.30}$$

$$d\varphi_t(x) = F_\lambda(\varphi_t(x)\theta_x(\varphi_t(x)))dt + g(\varphi_t(x))\hat{\circ}\, dw_t,\ \varphi_T(x) = x \tag{3.31}$$

$$d_s\varphi_{ts}^{-1}(x) = F_\lambda(\varphi_{ts}^{-1}(x)\ ,\theta_x(\varphi_{ts}^{-1}(x)))ds + g(\varphi_{ts}^{-1}(x)) \circ dw_s, \varphi_{tt}^{-1}(x) = x$$

$$t \le s \le T: \qquad \varphi_t^{-1}(x) = \varphi_{tT}^{-1}(x) \tag{3.32}$$

It follows from [3,p.39] that $\varphi_t^n(x) \xrightarrow{P} \varphi_t(x)$ uniformly in probability on compacts of $[0,T] \times \mathbf{R}^d$. $\nu_t(x)$ is given by integrating the same expression as in (3.30) but with $\varphi_t(x)$ instead of $\varphi_t^n(x)$. We used the forward equation (3.32) to get the inverse of the backward stochastic flow $\varphi_t(x)$. It follows now from $W^n(t,(\xi_t^n)^{-1}(x)) = V^{o,n}(t,x)$, (3.25), (3.26), (3.27) and the uniqueness of the solution of the stochastic PDE (3.5) that we must have

$$W(t,\xi_t^{-1}(x)) = V^o(t,x).$$

For all this reasoning to hold we need to ensure that both (3.5) and (3.4) have a global solution (the former holds if $\varphi_t(x)$ is a global flow of diffeomorphisms a.s.). The following lemmas give conditions for the existence of global solutions for nonlinear stochastic PDE's (3.9).

<u>**Lemma 3.1**</u>

Consider the backward nonlinear SPDE (3.9) and assume (a′), (e) and (f). Then (3.9) has a unique global solution

$$V^0(t,x,\omega) = \theta(x) + F(0,\theta_x(0))(T-t) + G(0,\theta_x(0))[w(T,\omega) - w(t,\omega)]$$

i.e. $V^0(t,x,\omega)$ is separable.

<u>**Proof**</u> (f) is equivalent to $\chi_t(x) = \theta_x(\varphi_t(x))$, $t \in [0,T]$ where $\chi_t(x), \varphi_t(x)$ are stochastic characteristics given by (3.10), (3.12). Indeed applying Ito's backward rule [14, p.255] we see that $(\varphi_t(x), \chi_t(x)) \in L$ for all $t \in [0,T]$ implies

$$\begin{aligned} d\chi_t(x) &- \theta_{xx}(\varphi_t(x))d\varphi_t(x) = -F_\varphi(\varphi_t(x),\chi_t(x))dt \\ &- G_\varphi(\varphi_t(x),\chi_t(x))\hat{o}\, dw_t - \theta_{xx}(\varphi_t(x))F_\chi(\varphi_t(x),\chi_t(x))dt - \theta_{xx}(\varphi_t(x)) \\ &\times G_\chi(\varphi_t(x),\chi_t(x))\hat{o}\, dw_t = \{F(\varphi,\chi),\chi-\theta_x(\varphi)\}\Big|_{\substack{\varphi=\varphi_t(x)\\ x=x_t(x)}} dt \\ &+ \{G(\varphi,\chi),\chi-\theta_x(\varphi)\}\Big|_{\substack{\varphi=\varphi_t(x)\\ \chi=\chi_t(x)}} \hat{o}\, dw_t = 0 \quad . \end{aligned}$$

To see that (f) implies $(\varphi_t(x),\chi_t(x)) \in L$ for all $t \in [0,T]$ we make the change of coordinates

$$T : \begin{cases} \bar\varphi = \varphi \\ \bar\chi = \chi - \theta_x(\varphi) \end{cases}$$

Indeed T is a change of coordinates as

$$\frac{\partial T(\varphi,\chi)}{\partial(\varphi,\chi)} = \begin{bmatrix} I_d & 0_d \\ -\theta_{xx}(\varphi) & I_d \end{bmatrix}$$

in the new coordinates (f) is equivalent to $(\bar f)$:

$$T(L) = \bar L = \{(\bar\varphi, \bar\chi) \in \mathbb{R}^{2d} | \bar\chi = 0\}$$

$$(\bar f) \begin{cases} \bar X_F(\bar\chi_i)|_{\bar L} = 0 \\ \bar X_{G^\ell}(\bar\chi_i)|_{\bar L} = 0 \end{cases}$$

where $\bar X_F(\bar\varphi,\bar\chi) = \dfrac{\partial T(T^{-1}(\bar\varphi,\bar\chi))}{\partial(\varphi,\chi)} X_F\,(T^{-1}(\bar\varphi,\bar\chi)),\ \bar X_{G^\ell}(\bar\varphi,\bar\chi) = \dfrac{\partial T(T^{-1}(\bar\varphi,\bar\chi))}{\partial(\varphi,\chi)}$

$\times\ X_{G^\ell}(T^{-1}(\bar\varphi,\bar\chi))$ and $X_F = F_\chi \dfrac{\partial}{\partial\varphi} - F_\varphi \dfrac{\partial}{\partial\chi}$, $X_{G^\ell} = G^\ell_\chi \dfrac{\partial}{\partial\varphi} - G^\ell_\varphi \dfrac{\partial}{\partial\chi}$;

$\ell = 1,\dots,p$; $i = 1,\dots,d$

The stochastic characteristic equations in the new coordinates become with our notations:

$$\begin{bmatrix} d\bar\varphi_t(x) \\ d\bar\chi_t(x) \end{bmatrix} = \begin{bmatrix} \bar X^1_F \\ \bar X^2_F \end{bmatrix} (\bar\varphi_t(x),\bar\chi_t(x))\,dt + \sum_{\ell=1}^{p} \begin{bmatrix} \bar X^1_{G^\ell} \\ \bar X^2_{G^\ell} \end{bmatrix} (\bar\varphi_t(x),\bar\chi_t(x)) \circ dw^\ell_t$$

$$:= \bar{X}_F\,(\bar{\varphi}_t(x),\bar{\chi}_t(x))\,dt + \sum_{\ell=1}^{p} \bar{X}_{G^\ell}(\bar{\varphi}_t(x),\bar{\chi}_t(x))\circ dw_t^\ell \;;\; \bar{\varphi}_T(x)=x,\bar{\chi}_T(x)=0 \tag{3.33}$$

and $(\bar{f}\,)$ implies

$$\bar{X}_F^2(\bar{\varphi}_t(x),0)=0, \quad \bar{X}^2_{G^\ell}(\bar{\varphi}_t(x),0)=0;\;\; \ell=1,\dots,p.$$

Under the assumptions (a) made on f, g_ℓ, θ (3.33) has for all $x \in \mathbb{R}^d$ a unique solution and this is $(\bar{\varphi}_t^*(x), \bar{\chi}_t^*(x)=0)$, $t \in [0,T]$: where $\bar{\varphi}_t^*(x)$ is the unique solution of

$$d\bar{\varphi}_t^*(x)=\bar{X}^1_F(\bar{\varphi}_t^*(x),0)dt + \sum_{\ell=1}^{p} \bar{X}^2_{G^\ell}(\bar{\varphi}_t^*(x),0),\circ dw_t^\ell \;;\; \bar{\varphi}_T^*(x)=0.$$

As $\bar{\chi}_t^*(x)=\chi_t(x)-\theta_x(\varphi_t(x))=0$, we obtain $(\varphi_t(x),\chi_t(x)) \in L$ for all $t \in [0,T]$.

The geometric interpretation of (f) is that in every point of the Lagrangian submanifold $L=\{(\varphi,\chi)\in \mathbb{R}^{2d} \,|\, \chi=\theta_x(\varphi)\}$ the drift and diffusion Hamiltonian vector fields

$$X_F = F_\chi \frac{\partial}{\partial\varphi} - F_\varphi \frac{\partial}{\partial\chi}$$

$$X_{G^\ell} = G^\ell_\chi \frac{\partial}{\partial\varphi} - G^\ell_\varphi \frac{\partial}{\partial\chi} \quad .$$

are tangent to L (we use the vectorial Poisson bracket notation from the LQG Example)

$$-\{F,\chi_i-\theta_{x_i}(\varphi)\}|_L =: X_F(\chi_i-\theta_{x_i}(\varphi))|_L = 0$$

$$\{G^\ell, \chi_i-\theta_{x_i}(\varphi)\}|_L = X_{G^\ell}(\chi_i-\theta_{x_i}(\varphi))|_L = 0 \;\;; i=1,\dots,d$$

so that as the stochastic characteristic system (3.10), (3.11) (seen as a stochastic Hamiltonian system) starts at t=T on L ($\chi_T(x) - \theta_x(\varphi_T(x)) = 0$ for all $x \in \mathbf{R}^d$) it will remain there for all $0 \le t \le T$ a.s.. L is thus an a.s. invariant submanifold for (3.10), (3.11). As a consequence $\varphi_t(x)$ is the flow of (3.31) which is an SDE with Lipschitz coefficients due to (a) so that $\varphi_t(x)$ is a (global) flow of diffeomorphisms a.s. for $t \in [0,T]$[12]. Thus $V^0(t,x) = \nu_t \circ \varphi_t^{-1}(x)$ is the global solution of (3.9). But more can be said about $V^0(t,x)$. As

$$\frac{\partial V^0}{\partial x}(t,x) = \chi_t \circ \varphi_t^{-1}(x) = \theta_x(x).$$

$V^0(t,x)$ is separable

$$V^0(t,x) = \theta(x) + \xi(t,\omega)$$

To find $\xi(t,\omega)$, a backward random field with the differential

$$d\xi(t,\omega) = \xi^1(t)dt + \sum_{\ell=1}^{p} \xi_\ell^2(t)\, \hat{d}w_t^\ell \tag{3.34}$$

we substitute (3.34) in (3.9):

$$[\xi^1(t) + F(x,\theta_x(x))]dt + \sum_{\ell=1}^{p} [\xi_\ell^2(t) + G^\ell(x,\theta_x(x))]\hat{o}\, dw_t^\ell \equiv 0.$$

Due to (f) $F(x,\theta_x(x)) = F(0,\theta_x(0))$, $G^\ell(x,\theta_x(x)) = G^\ell(0,\theta_x(0))$ so that $\xi(t,\omega) = F(0,\theta_x(0)(T\text{-}t)$

$$+\sum_{\ell=1}^{P} G^\ell(0,\theta_x(0))[w^\ell(T,\omega) - w^\ell(t,\omega)].$$

<u>**Lemma 3.2**</u> Consider the backward nonlinear SPDE (3.9) and assume (a), (c'), (f') and (f''). Then (3.9) has a unique C^2 global solution

$$V^0(t,x) = \nu_t \circ \varphi_t^{-1}(x)$$

with $\varphi_t^{-1}(x)$ the inverse flow of (3.16) and $\nu_t(x)$ given by (3.8) in which $\varphi_t(x)$ of (3.16) is substituted.

Proof (f′) and (f″) imply that for i=1,...,d.

$$F_{\chi_i}(\varphi_t(x),\chi_t(x)) = F_{\chi_i}(x,\theta_x(x)) = F_{\chi_i}(0,\theta_x(0)).$$

i.e. $F_\chi(\varphi_t(x),\chi_t(x))$ are d prime integrals (conservation laws) (see Arnold [1], Bismut [3]). As a result the first stochastic characteristic equation is an SDE (3.16) ("decoupled" from $\chi_t(x)$) and $\varphi_t(x)$ is a global flow of diffeomorphisms almost surely because of (a′).

Generalizations of Lemmas 3.1 and 3.2 leading to Theorems 3.3 and 3.4 are made by allowing the conservation law to be time dependent and respectively both time dependent and random so that it is a random field having a certain backward differential.In this way (g) and (g′) imply that $\chi_t(x)=\vartheta(t,\varphi_t(x))$ and respectively $\chi_t(x) = \vartheta(t,\varphi_t(x),\omega)$ *(the differential of the random field being* (3.20) so that again $\varphi_t(x)$ is given by an SDE ("decoupled" from $\chi_t(x)$) which in the case of random field conservation laws has random coefficients. In this case the global diffeomorphic property of $\varphi_t(x)$ follows from [12,§4.6] . (a′) and the local integrability w.r.t. time a.s. of the derivatives in x of the random field assumed in Theorem 3.4.

We need a result connecting the existence of a global C^2 solution for (3.9) to the existence of a global $C^{1,2}$ solution for (3.4) so that the latter is implied by the former. We will prove this when (f) is assumed. The cases when any of the assumptions (f′)-(f″), (g) or (g′) are made are treated similarly.

Proposition 3.1 Assume (a′), (e) and (f). Then (3.4) has a unique $C^{1,2}$ global solution given by $W(t,\eta) = \bar{\gamma}_t \circ \bar{\psi}_t^{-1}(\eta)$ where $\bar{\psi}_t^{-1}(\eta)$ is the inverse flow of

$$\frac{d\bar{\psi}_t(\eta)}{dt} = F_\delta(\xi_t(\bar{\psi}_t(\eta)),\ \theta_x^T(\bar{\psi}_t(\eta))\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}(\bar{\psi}_t(\eta))),\ \bar{\psi}_T(\eta) = \eta \tag{3.35}$$

and $\bar{\gamma}_t$ is given by (3.13) with $\bar{\delta}_t=\theta_x(\bar{\psi}_t(\eta))$ and $\psi_t=\bar{\psi}_t(\eta)$ substituted in the equation for γ_t.

Proof We will show that the characteristic sub-system for (3.4) made of the first and second equations of (3.13) has a unique global solution a.s. given by $(\overline{\psi}_t(\eta), \theta_x(\overline{\psi}_t(\eta)))$. First due to the assumptions on f,g, θ and ϕ we see that (3.15) has unique solution and (3.35) has unique global solution generating a global flow of diffeomorphisms a.s. due to $\xi_t(x)$ being a.s. flow of C^1 global diffeomorphisms and to the inequality(see Ocone and Pardoux [15])

$$\sup_{t \le T} \left| \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(x) \right| \le \zeta(\delta)(1+|x|^2)^{\delta} \qquad \text{for every } \delta > 0 \tag{3.36}$$

where $\zeta(\delta)$ are L^p-bounded random variables for $p \ge 1$. (3.36) plugged in the expression of $\overline{F}^{\omega}_{\delta}$ (see (3.13)) ensures this has a.s. linear growth. Showing that (3.13) has the solution $(\overline{\psi}_t(\eta), \overline{\delta}_t = \theta_x(\overline{\psi}_t(\eta)))$ amounts to checking that (f) implies that $\overline{\delta}_t = \theta_x(\overline{\psi}_t(\eta))$ satisfies (we omit η):

$$\frac{d\overline{\delta}_t}{dt} = -F_t(\xi_t(\overline{\psi}_t), \overline{\delta}_t \left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\overline{\psi}_t)) := -\overline{F}^{\omega}_{\psi}(t, \overline{\psi}_t, \overline{\delta}_t).$$

$$\overline{\delta}_T = \theta_x(\xi_T(\eta)) \frac{\partial \xi_T}{\partial x}(\eta)$$

that is, using the summation convention,

$$F_{\psi_i}\Big|_{\overline{\psi}_t, \overline{\delta}_t} + \theta_{x_j x_i}(\overline{\psi}_t) F_{\delta_j}\Big|_{\overline{\psi}_t, \overline{\delta}_t} = 0 \tag{3.37}$$

We first check the terminal condition

$$\overline{\delta}_T = \theta_x(\xi_T(\eta)) \frac{\partial \xi_T}{\partial x}(\eta) = \theta_x(\overline{\psi}_T(\eta)) = \theta_x(\eta)$$

This holds because (f) is equivalent to

$$0 = \{x_j g_{j\ell}(\varphi),\ x_k - \theta_{x_k}(\varphi)\}\Big|_I = \frac{\partial}{\partial x_k}\left(\theta_{x_j} g_{j\ell}\right)(\varphi)$$

$\forall, \varphi \in \mathbf{R}^d$; $\ell=1,\dots,p$; $k=1,\dots,d$ which implies that for all $\eta \in \mathbf{R}^d$, $t \in [0,T]$:

$$\theta_x(\eta)=\theta_x(\xi_t(\eta))\frac{\partial \xi_t}{\partial x}(\eta) = \theta_x\,(\xi_T(\eta))\,\frac{\partial \xi_T}{\partial x}(\eta) \quad \text{a.s.} \tag{3.38}$$

This can be seen by applying Ito's rule

$$d(\theta_{x_j}(\xi_t(\eta))\Big(\frac{\partial \xi_t}{\partial x}\Big)_{ji}(\eta)) = \Big(\frac{\partial \xi_t}{\partial x}\Big)_{ki}(\eta)\,\theta_{x_j x_k}(\xi_t(\eta))g_{j\ell}(\xi_t(\eta)) \circ dw_t^\ell + \theta_{x_j}(\xi_t(\eta))\Big(\frac{\partial}{\partial x_k}\,g_{j\ell}\Big)\Big(\frac{\partial \xi_t}{\partial x}\Big)_{ki}(\eta)$$

$$\circ\, dw_t^\ell = \Big(\frac{\partial \xi_t}{\partial x}\Big)_{ki}(\eta)\,\frac{\partial}{\partial x_k}\Big(\theta_{x_j} g_{j\ell}\Big)(\xi_t(\eta)) \circ dw_t^\ell$$

where we have used

$$d\,\Big(\frac{\partial \xi_t}{\partial x}\Big)_{ji}(\eta) = \frac{\partial}{\partial x_k}\,g_{j\ell}(\xi_t(\eta))\,\Big(\frac{\partial \xi_t}{\partial x}\Big)_{ki}(\eta) \circ dw_t^\ell.$$

As a consequence we also have

$$\theta_{x_i}(\eta)\,\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}_{i\ell}(\eta) = \theta_{x_\ell}(\xi_t(\eta))\,. \tag{3.39}$$

We differentiate with respect to η_j

$$\theta_{x_j x_i}(\eta)\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}_{i\ell}(\eta) + \theta_{x_k}(\eta)\,\frac{\partial}{\partial x_j}\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}_{k\ell}(\eta) = \theta_{x_k x_\ell}(\xi_t(\eta))\,\Big(\frac{\partial \xi_t}{\partial x}\Big)_{kj}(\eta) \tag{3.40}$$

Remembering the definition of $F(\varphi,\lambda)$ in (3.9) we have the following relations between F_{ψ_i}, F_{δ_j} and F_{λ_k} where $F_{\psi_i} := F_{\psi_i}(\xi_t(\psi),\delta^T\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}(\psi))$ etc. and $F_{\varphi_k} := F_{\varphi_k}(\varphi,\lambda)\Big|_{\bar\varphi,\bar\lambda}$ with

$\bar\varphi\,(t,\omega,\psi,\delta)=\xi_t(\psi)$, $\bar\lambda\,(t,\omega,\psi,\delta) = \delta^T\,\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}(\psi)$:

$$F_{\psi_i} = F_{\varphi_k}\,\Big(\frac{\partial \xi_t}{\partial x}\Big)_{ki}(\psi) + F_{\lambda_k}\delta_\ell\,\frac{\partial}{\partial x_i}\,\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}_{\ell k}(\psi)$$

$$F_{\delta_j} = F_{\lambda_k}\,\Big(\frac{\partial \xi_t}{\partial x}\Big)^{-1}_{jk}(\psi)$$

which when introduced in (3.37) yield that proving the proposition now amounts to showing (we omit arguments of functions):

$$E:=\left(F_{\varphi_\ell}\left(\frac{\partial\xi_t}{\partial x}\right)_{\ell i}\right)\Big|_{\bar\psi_t,\bar\delta_t}+F_{\chi_k}\Big|_{\bar\psi_t,\bar\delta_t}\left(\theta_{x_\ell}(\bar\psi_t)\frac{\partial}{\partial x_i}\left(\frac{\partial\xi_t}{\partial x}\right)^{-1}_{\ell k}(\bar\psi_t)+\theta_{x_j x_i}(\bar\psi_t)\cdot\left(\frac{\partial\xi_t}{\partial x}\right)^{-1}_{jk}(\bar\psi_t)\right)$$

$$=0$$

But interchanging the summation indices in (3.40) $j\to i$, $\ell\to k$ and evaluating for $\eta=\bar\psi_t$ we get

$$E=\left(F_{\varphi_\ell}\left(\frac{\partial\xi_t}{\partial x}\right)_{\ell_i}+F_{\chi_k}\theta_{x_\ell x_k}\left(\frac{\partial\xi_t}{\partial x}\right)_{\ell i}\right)\Big|_{\bar\psi_t,\bar\delta_t}$$

$$=\{F(\varphi,\chi),\ \chi_\ell-\theta_{x_\ell}(\varphi)\}|_{\varphi_t^*,\chi_t^*}\times\left(\frac{\partial\xi_t}{\partial x}\right)_{\ell i}(\bar\psi_t)$$

where $\varphi_t^*:=\xi_t(\bar\psi_t)$, $\chi_t^*:=\bar\delta_t\left(\frac{\partial\xi_t}{\partial x}\right)^{-1}(\bar\psi_t)$.

Due to (3.39) and $\bar\delta_t=\theta_x(\bar\psi_t)$

$$\chi_t^*=\theta_x(\bar\psi_t)\left(\frac{\partial\xi_t}{\partial x}\right)^{-1}(\bar\psi_t)=\theta_x(\xi_t(\bar\psi_t))=\theta_x(\varphi_t^*).$$

for all $t\in[0,T]$ so that $(\varphi_t^*,\chi_t^*)\in L$ a.s. for all $t\in[0,T]$ and (f) implies

$$\{F(\varphi,\chi),\ \chi_\ell-\theta_{x_\ell}(\varphi)\}|_{(\varphi_t^*,\chi_t^*)\in L}=0$$

which proves the characteristic system (3.13) has the unique global solution $(\bar\psi_t(\eta),\bar\delta_t(\eta)=\theta_x(\bar\psi_t(\eta)))$ and thus (3.4) has the unique global solution $W((t,\eta)=\gamma_t\circ\bar\psi_t^{-1}(\eta)$ where $\bar\psi_t^{-1}(\eta)=\bar\psi_{tT}^{-1}(\eta)$ with $\bar\psi_{tT}^{-1}(\eta)$ given by the forward random equation

$$\frac{d\bar\psi_{ts}^{-1}(\eta)}{ds}=F_\delta(\xi_s(\bar\psi_{ts}^{-1}(\eta)),\ \theta_x^T(\bar\psi_{ts}^{-1}(\eta))\left(\frac{\partial\xi_s}{\partial x}\right)^{-1}(\bar\psi_{ts}^{-1}(\eta))),$$

$$\bar{\psi}_{ti}^{-1}(\eta) = \eta \qquad : \qquad t \leq s \leq T \tag{3.41}$$

Interpretation of the result

The Hamiltonian geometric interpretation of this result is that the solution decomposition formula (1.2) leads to defining a canonical point transformation (see [3, p. 339], [1, p. 239], [16. p.82])

$$\varphi = \xi_t(\psi)$$
$$\lambda = \delta^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi)$$

between the random Hamiltonian system

$$\begin{cases} \dot{\psi}_t = \bar{F}_{\delta}^{\varphi}(t,\psi_t,\delta_t) & : \qquad \psi_T = \eta \\ \dot{\delta}_t = -\bar{F}_{\psi}^{\varphi}(t,\psi_t,\delta_t) & : \qquad \delta_T = \theta_x(\xi_T(\eta))\dfrac{\partial \xi_T}{\partial x}(\eta) \end{cases} \tag{3.42}$$

and the stochastic Hamiltonian system

$$\begin{cases} d\varphi_t = F_\lambda(\varphi_t,\lambda_t)dt + G_\lambda(\varphi_t,\lambda_t)\circ dw_t \; ; \; \varphi_T = x \\ d\lambda_t = -F_\varphi(\varphi_t,\lambda_t)dt - G_\varphi(\varphi_t,\lambda_t)\circ dw_t ; \; \lambda_T = \theta_x(x) \end{cases} \tag{3.43}$$

Being canonical the transformation preserves Poisson brackets

$$\{F(\varphi,\lambda),\ \lambda - \theta_x(\varphi)\}_{\varphi,\lambda} = \{F(\xi_t(\psi),\delta^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi)),\delta^T\left(\frac{\partial \xi_t}{\partial x}\right)^{-1}(\psi) - \theta_x(\xi_t(\psi))\}_{\psi,\delta}$$

which can be shown following the computations from the classical Hamiltonian case from Rund [17, p. 86-93] for our case. This way (3.43) having the Lagrangian invariant submanifold L implies that (3.42) will also have L as invariant submanifold via (f). One can proceed like this using the theory of canonical transformations for stochastic Hamiltonian systems of Bismut [3, p.339] to develop a theory of canonical point transformation between stochastic Hamiltonian systems and the robust random Hamiltonian systems associated with them. Proving the convergence result (3.26) when (f) is assumed means in the light of Proposition 3.1 proving that for each $t \in [0,T]$

$$\bar{\psi}_t^n(\eta) \xrightarrow{P} \bar{\psi}_t(\eta)$$

$$(\bar{\psi}_t^n)^{-1}(\eta) \xrightarrow{P} \bar{\psi}_t^{-1}(\eta)$$

uniformly in η on compacts where

$$\frac{d\bar{\psi}_t^n(\eta)}{dt} = F_\delta(\xi_t^n(\bar{\psi}_t^n(\eta)),\ \theta_x^T(\bar{\psi}_t^{-1}(\eta))\Big(\frac{\partial\xi_t^{-1}}{\partial x}\Big)^{-1}(\bar{\psi}_t^n(\eta))),\ \bar{\psi}_T^n(\eta) = \eta.$$

This is proved in the same way as (2.35). The random characteristic system (3.13) is "decoupled" $\Big(\bar{\delta}_t^n = \theta_x(\bar{\psi}_t^n)\Big)$ under (f) and so for each $t \in [0,T]$ by continuity

$\bar{\gamma}_t^n(\eta) \xrightarrow{P} \bar{\gamma}_t(\eta)$ uniformly in η on compacts thus yielding (3.26) for each $t \in [0,T]$:

$$W^n(t,\eta) = \bar{\gamma}_t^n \circ (\bar{\psi}_t^n)^{-1}(\eta) \xrightarrow{P} W(t,\eta) = \bar{\gamma}_t \circ \bar{\psi}_t^{-1}(\eta)$$

uniformly in η on compacts. One approaches similarly the cases when any of the assumptions (f′) - (f″), (g) or (g′) are made in order to "decouple" the stochastic and the random characteristic systems and to ensure the global existence of the inverse of the flows of characteristics.

We turn now to the "cost of information" issue.

<u>**Proof of Theorem 3.5 and its corollaries**</u>

Averaging (3.5) and interchanging expectation with differentiation and integration (we use again regularity results of the type of Lemma 6.2.6 and Theorem 6.1.10 from [10] in our particular case) we obtain

$$\frac{\partial EV^0}{\partial t}(t,x) + [\frac{\partial}{\partial x}(EV^0)(t,x)]f(x,u^*(t,x)) - [\frac{\partial}{\partial x}(EV^0)(t,x)]f(x,u^*(t,x))$$

$$+ E\Big(\frac{\partial V^0}{\partial x}(t,x)f(x,\phi(V_x^0(t,x),x))\Big) + \frac{1}{2}tr\Big(\frac{\partial^2}{\partial x^2}(EV^0)(t,x)gg^T(x)\Big) = 0 \qquad (3.44)$$

$$EV^0(T,x) = \theta(x)$$

where we added and subtracted $\partial/\partial x(EV^0)(t,x)f(x,u^*(t,x))$.

Subtract (3.44) from (see c′)).

$$\frac{\partial V^*}{\partial t}(t,x) + \frac{\partial V^*}{\partial x}(t,x)f(x,u^*(t,x)) + \tfrac{1}{2}\mathrm{tr}\left(\frac{\partial^2 V^*}{\partial x^2}gg^T(x)\right) = 0$$
$$V^*(T,x) = \theta(x).$$

We obtain the PDE for the cost of perfect informationfor $\Delta(t,x) = V^*(t,x) - EV^0 t,x)$:

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x}f(x,u^*(t,x)) + \tfrac{1}{2}\mathrm{tr}\left(\frac{\partial^2 \Delta}{\partial x^2}gg^T(x)\right) + \frac{\partial}{\partial x}(EV^0)f(x,u^*(t,x))$$

$$- E\left(\frac{\partial V^0}{\partial x}f\left(x,\phi\left(\frac{\partial V^0}{\partial x},x\right)\right)\right) = 0 \tag{3.45}$$

$$\Delta(T,x) = 0.$$

We obtain (3.22) by representing probabilistically the solution of (3.45) and using the stochastic characteristics formula for $V^0(t,x)$:

$$\Delta(t,x)=E^{t,x}\int_t^T E[V_x^0(s,x)\Big(f(x,u^*(s,x)) - f(x,\phi(V_x^0(s,x),x)\Big)]\,\Big|_{x=x^*(s)}ds$$

Due to $x^*(s)$ being independent of $V_x^0(s,x)$ (the former is past adapted while the latter is future adapted) we get (3.22) using the characteristics representation $V_x^0(s,x)=\chi_s \circ \varphi_s^{-1}(x)$. To obtain the particular formula from the Corollary 3.1 we use $\frac{\partial V^0}{\partial x}(t,x) = \beta(t,x,\omega))$, $u^0(t,x) = \phi(\beta(t,x,\omega),x)$. (see Theorem 3.4). We prove next that assumption (f) (i.e. the Lagrangian submanifold is invariant) leads to zero cost of information. The case when (g) is assumed to hold (i.e. existence of a time varying deterministic conservation law) is proved similarly using generalized Poisson bracket computations (time is included as additional variable) instead of the usual Poisson bracket. As we have seen in Theorem 3.1 the anticipative optimal control is nonanticipative (feedback) $u^0(t,x)=\phi(\theta_x(x),x)$ and

$$EV^0(t,x)=\theta(x)+\theta_x(0)\Big(f(0,\phi(\theta_x(0),0))-\tfrac{1}{2}g_xg(0)\Big)(T-t)$$

so that $EV^0_x=V^0_x=\theta_x(x)$ and $E\min_{u\in\mathcal{U}}\{V^0_xf(x,u)\}=\min_{u\in\mathcal{U}}\{(EV^0_x)f(x,u)\}$. By averaging the backward Ito form of (3.15) we see that EV^0_x satisfies the second order parabolic PDE of stochastic dynamic programming (2.7') being thus equal to $V^*(t,x)$ by unicity so that $\Delta(t,x)=0$. The same happens when $u^0(t,x)=\phi(\beta(t,x),x)$, $EV^0_x=V^0_x=\beta(t,x)$ (see theorem 3.3) because again the gradient of the pathwise value function is non random due to the existence of a deterministic conservation law . This is not the case when a random conservation law exists (Theorem 3.4) or when (f'),(f'') are assumed to hold and in this cases $\Delta(t,x)\neq 0$ and *the optimal control is anticipative.*

3.3 Example: Anticipative LQG

Consider the anticipative LQG problem (first considered in [5] using extension by continuity):

$$dx_t=(Ax_t+Bu_t)dt+Cdw_t$$

$$\inf_{u\in\mathcal{A}} E\left[\int_0^T (x_t^TQx_t+u_t^TRu_t)dt+x_T^TFx_T\right] \;;\; Q\,,\, F\geq 0 \;\;,\; R>0$$

Using the extension of our results to the case with integral cost term we obtain the HJB SPDE of LQG which is a quadratic nonlinear SPDE

$$dV^o+[V_x^oAx-(1/4)V_x^oBR^{-1}B^T(V_x^o)^T+x^TQx]dt+V_x^oC\circ dw_t=0$$

$$V^o(T,x)=x^TFx$$

and the optimal anticipative control in selector form $u^o(t,x,\omega)=-(1/2)R^{-1}B^TV_x^o$. The characteristics are(using the Ito backward differential notation [11, p.255]):

$$d\varphi_t=[A\varphi_t-\;(1/2)BR^{-1}B^T\lambda_t]dt-C\hat{d}w_t \;\;:\varphi_T=x$$

$$\frac{d\lambda_t}{dt}=-2Q\varphi_t-A^T\lambda_t \;:\; \lambda_T=2Fx$$

There exists a random conservation law $\lambda_t(x)=2S_t\varphi_t(x)+2\bar{\beta}_t(\omega)$ where S_t is the solution of the differential matrix Riccati equation

$$-\frac{dS_t}{dt}=S_tA+A^TS_t+Q-S_tBR^{-1}B^TS_t\ ,\ S_T=F$$

and

$$d\bar{\beta}_t(\omega)=\beta_1(t,\omega)dt+\beta_2(t,\omega)\circ dw_t\ ,\ \bar{\beta}_T(\omega)=0$$

$$-2\frac{dS_t}{dt}\varphi-2\beta_1(t,\omega)+\{\chi-2S_t\varphi-2\bar{\beta}_t,\chi^TA\varphi-\frac{1}{4}\chi^TBR^{-1}B^T\chi+\varphi^TQ\varphi\}|_{(\varphi,\chi)\in L_t(\omega)}=0$$

$$-2\beta_2(t,\omega)+\{\chi-2S_t\varphi-2\bar{\beta}_t,\chi^TC\}|_{(\varphi,\chi)\in L_t(\omega)}=0$$

$$L_t(\omega)=\{(\varphi,\chi)\in\mathbb{R}^{2d}|\ \chi=2S_t\varphi+2\bar{\beta}_t(\omega)\}$$

We can comment again that although the coefficients of the SPDE and SDE are not bounded with bounded derivatives the existence of an affine conservation law leading to an affine first characteristic equation ,decoupled from the second one, yields a global solution .It is important to point here that such a Hamiltonian mechanics point of view already led to interesting re-derivations of the Kalman filter by Bensoussan,Bismut,Mitter [3, p.356].The value function and the optimal anticipative control are :

$$V^o(t,x)=x^TS_tx+2\bar{\beta}_t(\omega)x+\gamma_t(\omega)\ ,\ u^o(t,x,\omega)=-R^{-1}B^T(S_tx+\bar{\beta}_t(\omega))$$

$$d\bar{\beta}_t=-(A^T-S_tBR^{-1}B^T)\bar{\beta}_tdt-S_tC\hat{d}w_t\ ,\ \bar{\beta}_T=0$$

$$d\gamma_t=\bar{\beta}_t^TBR^{-1}B^T\bar{\beta}_tdt-2\bar{\beta}_t^TC\hat{d}w_t\ ,\ \gamma_T=0$$

It is interesting to see that Riccati matrix differential equation appears in the conservation law formula. The cost of perfect information is (see also [5]) :

$$\Delta(t,x) = \int_t^T \mathrm{tr}(U_t B R^{-1} B^T) dt$$

where U_t is the symmetric nonnegative definite solution of the Lyapunov equation

$$\frac{dU_t}{dt} = (A^T - S_t B R^{-1} B^T) U_t + U_t (A - B R^{-1} B^T S_t) + S_t C C^T S_t^T \;:\; U_T = 0$$

3.4 Example : a nonlinear anticipative control problem

Consider next the scalar nonlinear anticipative optimal control problem :

$$dx_t=\left(\frac{\cos x_t+2}{2}+\frac{\sin x_t}{2(\cos x_t+2)^3}+u_t\right)dt+\frac{1}{\cos x_t+2}\,dw_t$$

$$\inf_{u\in\mathcal{A}} E\left[\frac{1}{2}\int_0^T u_t^2dt+\sin x_T+2x_T\right]$$

Using the obvious extension of our results to problems with integral costs we get the nonlinear HJB SPDE

$$dV^0+\left(\min_{u\in\mathcal{U}}\{V_x^0u+\tfrac{1}{2}u^2\}+V_x^0\frac{\cos x+2}{2}\right)dt+V_x^0\frac{1}{\cos x+2}\circ dw_t=0$$

$$V^0(T,x)=\sin x+2x$$

which for the minimizer $u^0(t,x,\omega)=-V_x^0(t,x,\omega)$ becomes

$$dV^0+\left(V_x^0\frac{\cos x+2}{2}-\frac{1}{2}(V_x^0)^2\right)dt+V_x^0\frac{1}{\cos x+2}\circ dw_t=0$$

We can check that (f) holds :

$$\{\lambda\frac{\cos\varphi+2}{2}-\tfrac{1}{2}\lambda^2,\ \lambda-\cos\varphi-2\}|_L=0$$

$$\{\lambda\frac{1}{\cos\varphi+2},\ \lambda-\cos\varphi-2\}|_L=0$$

for $(\varphi,\lambda)\in L=\{(\varphi,\lambda)\in\mathbf{R}^{2d}|\ \lambda-\cos\varphi-2=0\}$ and thus the Lagrangian submanifold L is invariant for the stochastic characteristics of this HJB SPDE

$$d\varphi_t=\left(\frac{\cos\varphi_t+2}{2}-\lambda_t\right)dt+\frac{1}{\cos\varphi_t+2}\circ dw_t\ \ ;\ \varphi_T=x$$

$$d\chi_t = \chi_t \frac{\sin\varphi_t}{2} dt - \chi_t \frac{\sin\varphi_t}{(\cos\varphi_t+2)^2} \circ dw_t \; ; \; \chi_T = \cos x + 2$$

i.e. $\chi_t = \cos\varphi_t + 2 \;\; \forall\, t \in [0,T]$ a.s. The optimal control is feedback $u^0(x) = -\cos x - 2$,

$V^0(t,x,\omega) = \sin x + 2x + w_T - w_t$ and $EV^0(t,x) = V^*(t,x)$, $\Delta(t,x)=0$ (zero cost of perfect information) as $\sin x + 2x$ is also the solution of the parabolic PDE of nonanticipative optimal control

$$\frac{\partial V^*}{\partial t} + \min_{u \in \mathcal{U}} \left\{ V_x^* u + \tfrac{1}{2}u^2 \right\} + V_x^* \left(\frac{\cos x + 2}{2} + \frac{\sin x}{2(\cos x + 2)^3} \right) + \frac{V_{xx}^*}{(\cos x+2)^2} = 0$$

$$V^*(T,x) = \cos x + 2$$

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